A theoretical approach to controlling quantum spin dynamics

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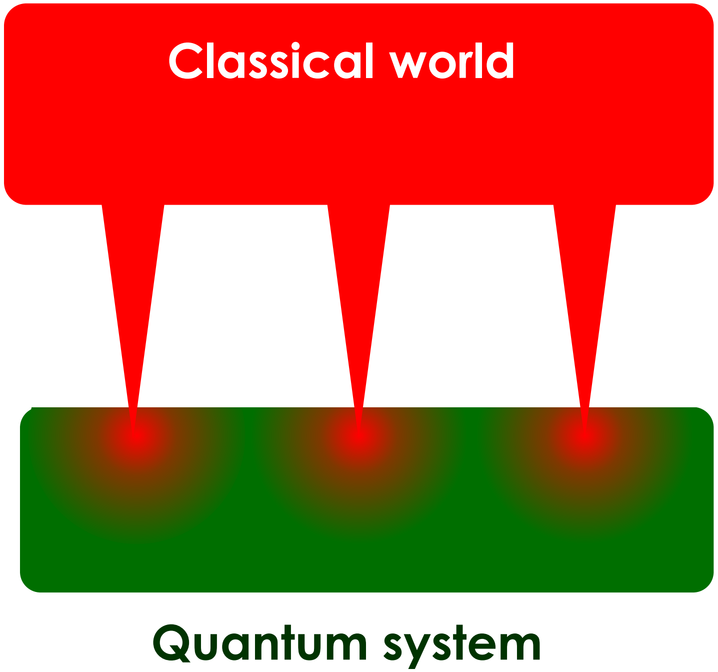
1. Introduction

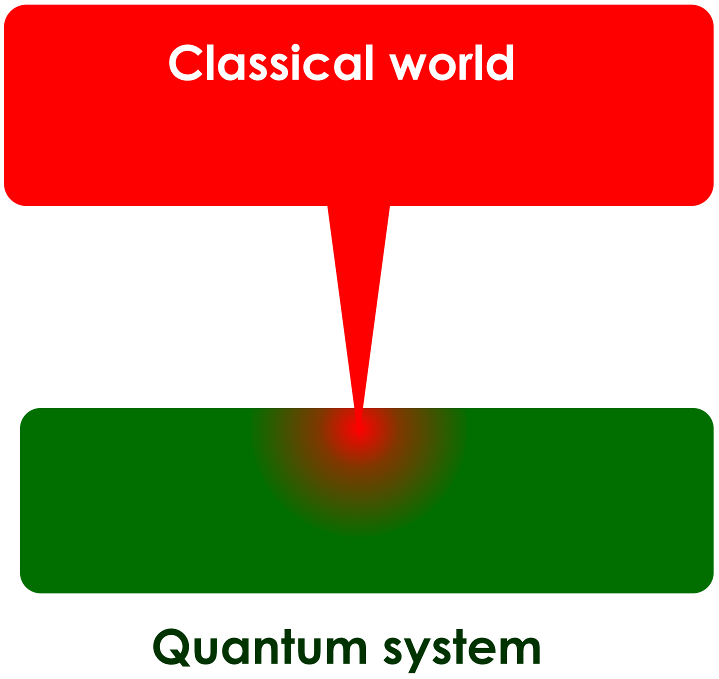
Goal: Full control over many-body quantum system

Applications: Quantum information processing, Control of chemical reactions, Energy conversion and storage



1. Introduction

Artificial control / access to quantum systems necessarily induce noise!

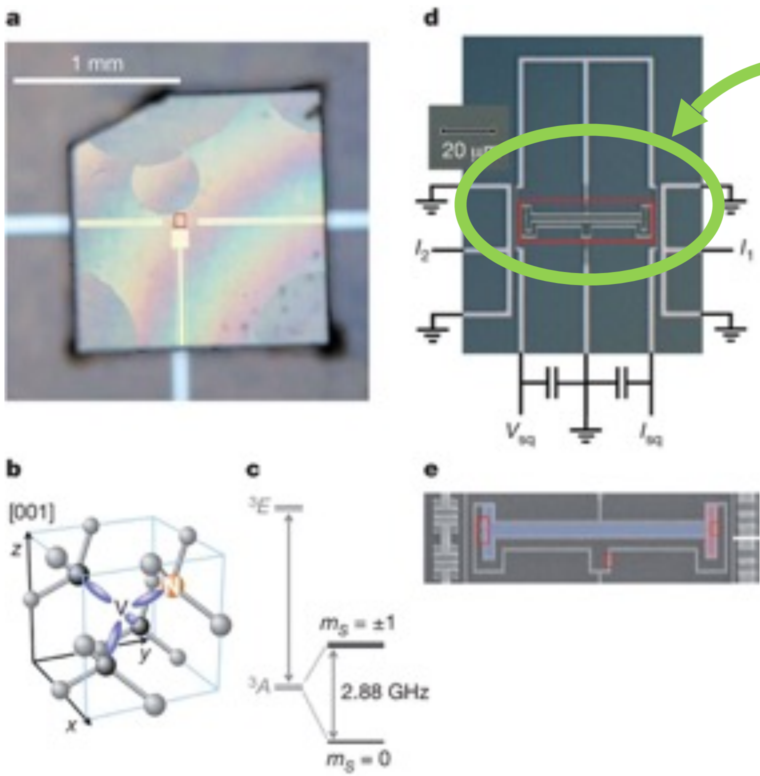


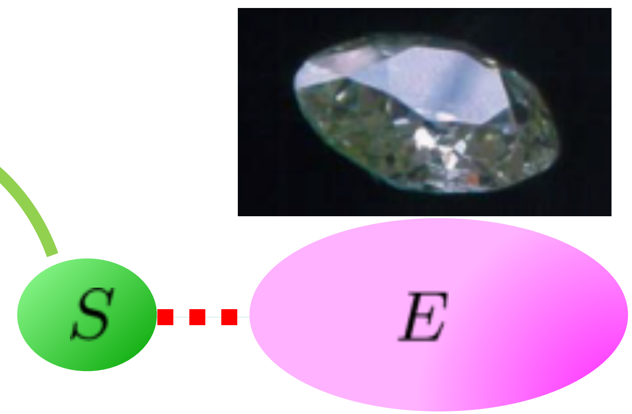
Want to minimise access to the quantum system of interest.

1. Introduction

An example of what we envisage...

Superconducting qubits coupled
with NV centres in diamond





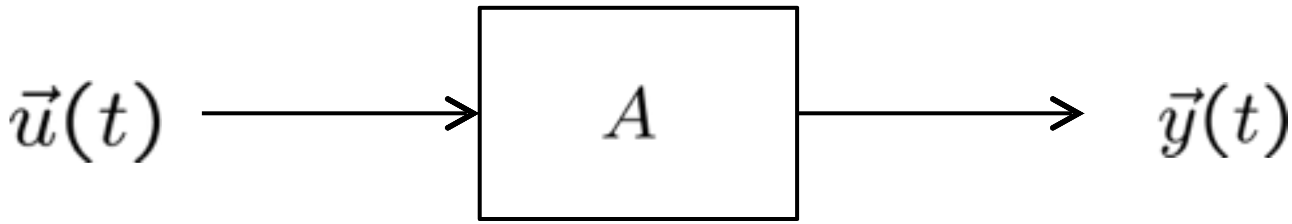
SC qubits spins at NV centres

Can a large quantum system be controlled through a small subsystem?

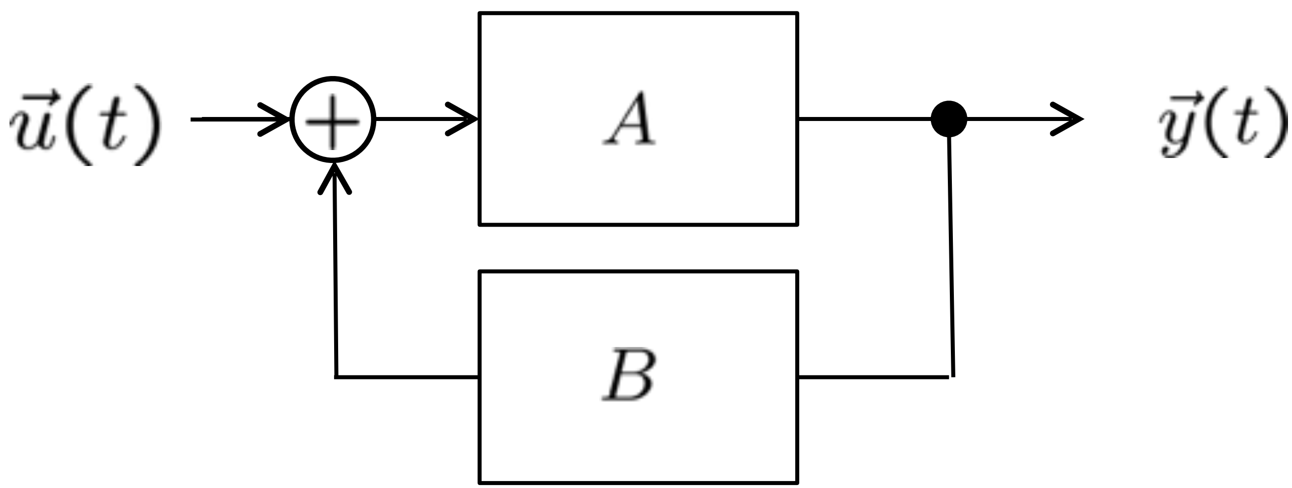
X. Zhu et al., Nature 478, 221 (2011)

Classical control

Open loop control

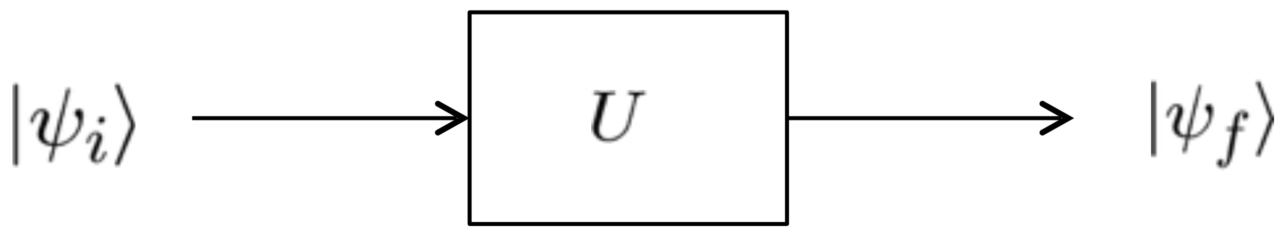


Closed loop control (Feedback)

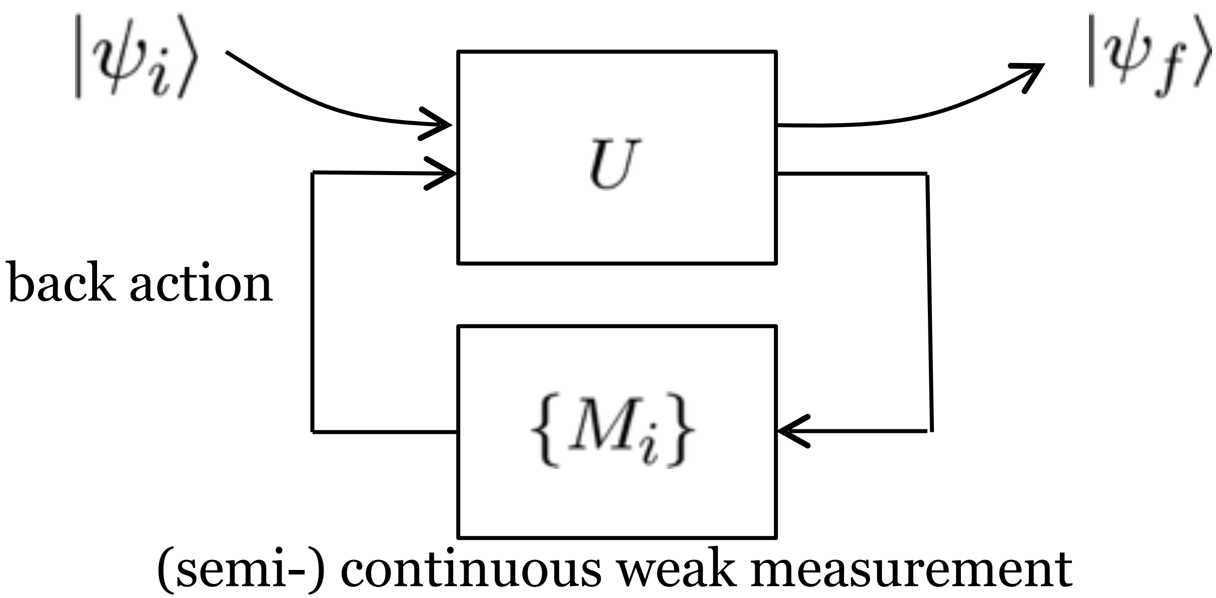


Quantum control

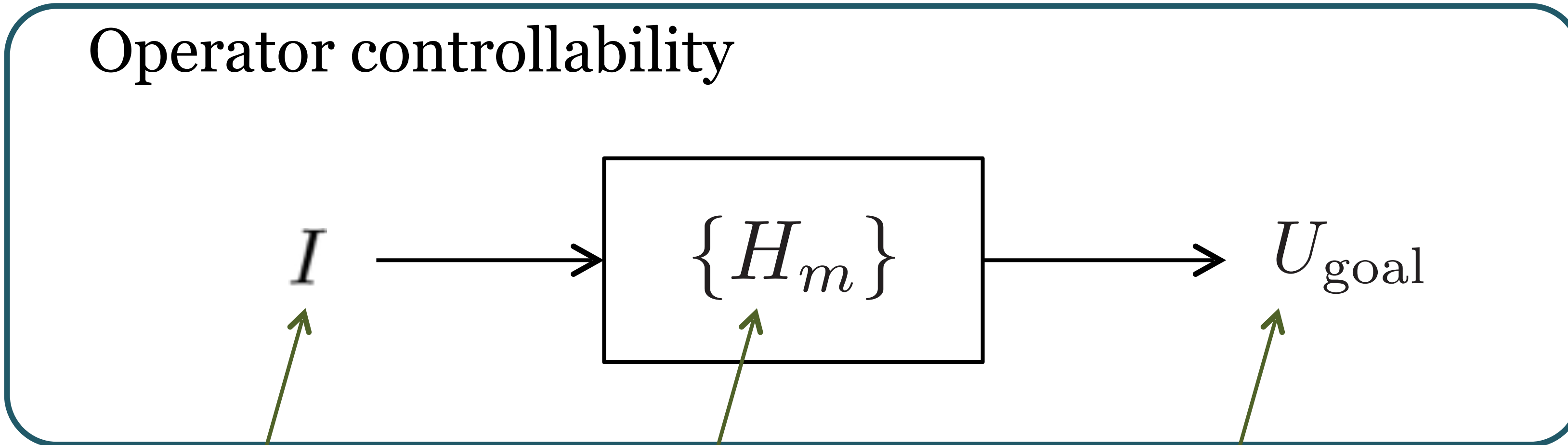
Open loop control



Closed loop control (Feedback)



Quantum control



the net operation at t=0
(identity operator)

$$I|\psi_{in}\rangle = |\psi_{in}\rangle$$

desired operation
(the goal unitary)

$$U_{\text{goal}}|\psi_{in}\rangle = |\psi_{\text{goal}}\rangle$$

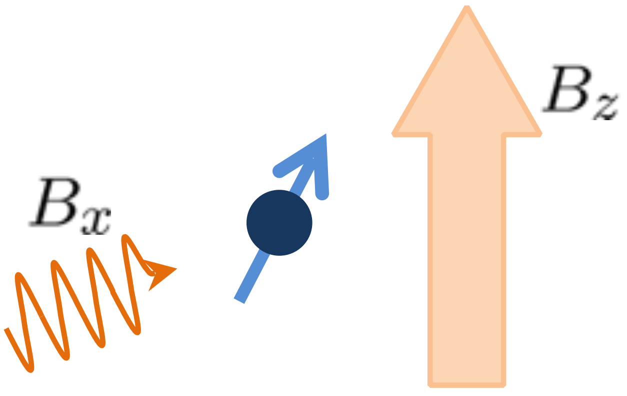
modulable Hamiltonians

\+ **un**modulable Hamiltonian

Quantum control

For example, the dynamics of a single spin in a static magnetic field B_z, with B_x as a controllable field is governed by

$$H(t) = \underbrace{B_z S_z}_{\text{unmodulable part}} + \underbrace{B_x(t) S_x}_{\text{modulable part}}$$



Quantum control

General form of Hamiltonian

$$H(t) = H_0 + \sum_m f_m(t) H_m$$

The dynamics is governed by the Schroedinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

The resulting change is equivalent to a unitary

$$|\psi(T)\rangle = U |\psi(0)\rangle$$

$$U = \mathcal{T} \exp\left(-i \int_0^T H(t) dt \right)$$

Quantum control

Question:

$$H(t) = H_0 + \sum_m f_m(t) H_m$$

What operations can we realise with the set $\{H_m\}$?

$(m = \{0, 1, 2, ..., M\})$

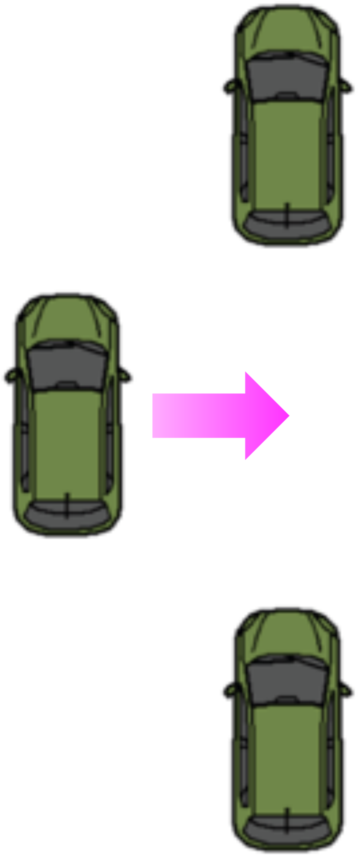
Answer:

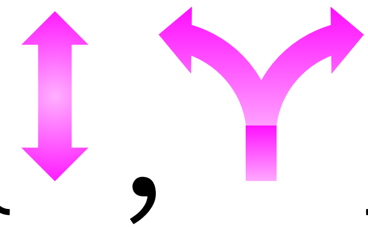
$\{e^{\mathcal{L}}\}$, where $\mathcal{L}$ is the *dynamical Lie algebra*

Ramakrishna et al., PRA51, 1995.

D'Alessandro, textbook, 2008.

Controllability with limited operations



Realisable with $\{$  $\}$?

metaphor by M. Murphy (and Prof T. Calarco) in Ulm

Controllability with limited operations

Lets simplify the elementary operations.

(still essentially the same)

1. Straight driving

$S(l)$

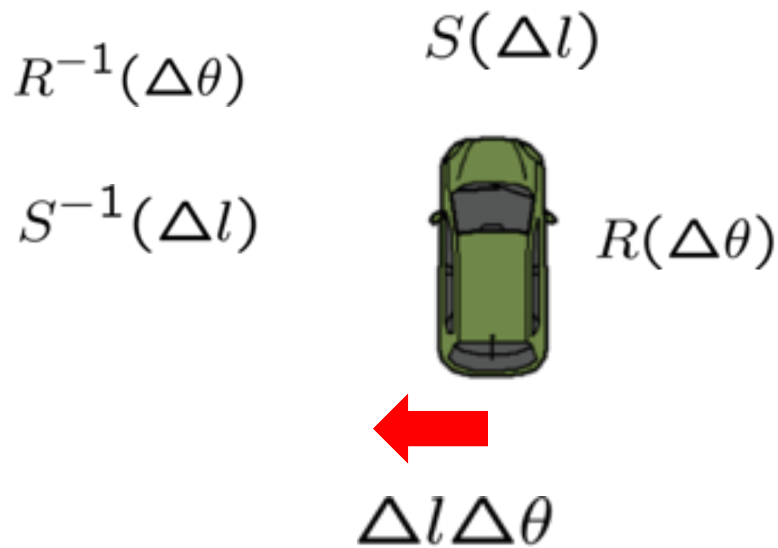
2. Rotation (of small angles)

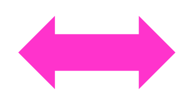
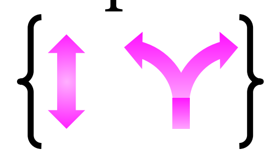
$R(\theta)$

Controllability with limited operations

The sequence of operations to realise a parallel transport

1. $R(\Delta\theta)$
2. $S(\Delta l)$
3. $R^{-1}(\Delta\theta)$
4. $S^{-1}(\Delta l)$



$S^{-1}(l)R^{-1}(\theta)S(l)R(\theta) =$ parallel transport 
which wasn't in the "modulable set", .

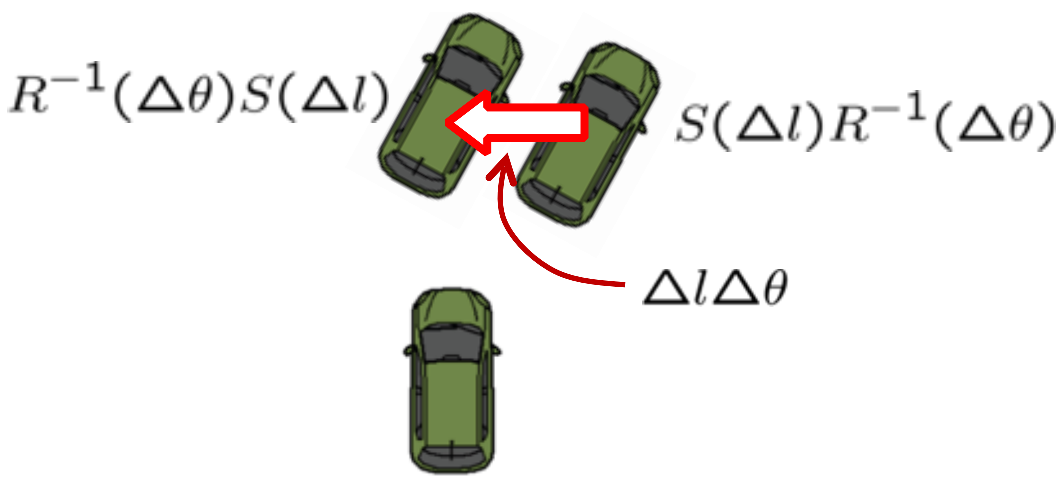
Controllability with limited operations

$$S^{-1}(l)R^{-1}(\theta)S(l)R(\theta) = \text{ parallel transport}$$

If $R(\theta)$ and $S(l)$ commute, i.e., $S(l)R(\theta) = R(\theta)S(l)$,

$$S^{-1}(l)\underline{R^{-1}(\theta)S(l)}R(\theta) = S^{-1}(l)S(l)R^{-1}(\theta)R(\theta)$$
$$= I \qquad \text{(zero net move)}$$

Difference between $R^{-1}(\Delta\theta)S(\Delta l)$ and $S(\Delta l)R^{-1}(\Delta\theta)$



Controllability with limited operations

$$S^{-1}(l)R^{-1}(\theta)S(l)R(\theta) = \text{ parallel transport}$$

If $R(\theta)$ and $S(l)$ commute, i.e., $S(l)R(\theta) = R(\theta)S(l)$,

$$\begin{aligned} S^{-1}(l)\underline{R^{-1}(\theta)S(l)}R(\theta) &= S^{-1}(l)S(l)R^{-1}(\theta)R(\theta) \\ &= I \qquad \text{(zero net move)} \end{aligned}$$

Noncommutativity of generators gives rise to a nontrivial operation(s).

Note: *Generators* are infinitesimal operations (roughly speaking). In quantum mechanics, Hamiltonians are generators.

Quantum control

Question:

$$H(t) = H_0 + \sum_m f_m(t) H_m$$

What operations can we realise with the set $\{H_m\}$?

$(m = \{0, 1, 2, ..., M\})$

Answer:

$\{e^{\mathcal{L}}\}$, where $\mathcal{L}$ is the *dynamical Lie algebra*

Ramakrishna et al., PRA51, 1995.
D'Alessandro, textbook, 2008.

Dynamical Lie algebra $\mathcal{L}$

Definition:

The set of generators obtained by taking commutators of the given Hamiltonians repeatedly and their real linear combinations.

e.g., $[iH_\alpha, iH_\beta],\ [iH_\alpha, [iH_\beta, iH_\delta]] + c \cdot iH_\gamma,$ etc.

<u>Single spin:</u>

If we can control X and Y, because $[iX, iY] = -2iZ$,

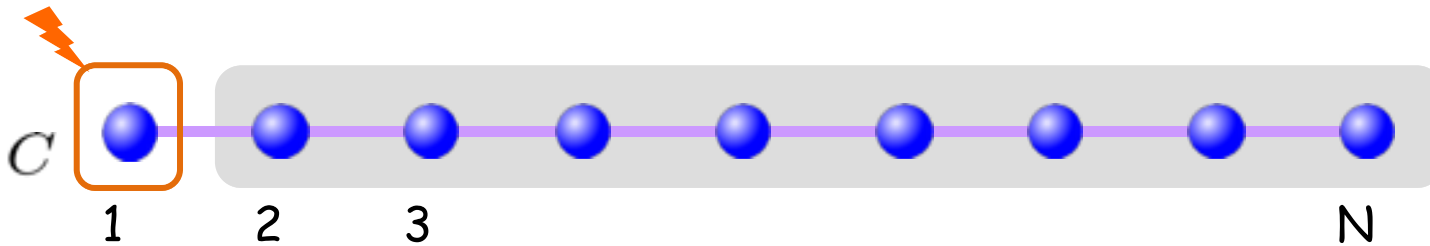
$$\mathcal{L} = \{iX, iY, iZ\} = \mathrm{su}(2)$$

Thus fully controllable. $\{X, Y, Z\}$: Pauli matrices

Cf. Euler angles $R_{\vec{n}}(\theta) = R_z(\alpha)R_y(\beta)R_z(\gamma)$

Dynamical Lie algebra $\mathcal{L}$

Heisenberg spin chain:



$$H = H^{\text{int}} + g\mu_B \vec{S}_1 \cdot \vec{B}_1(t)$$

$$H^{\text{int}} = \sum_n c_n (X_n X_{n+1} + Y_n Y_{n+1} + Z_n Z_{n+1})$$

The dynamical Lie algebra $\mathcal{L}$ coincides with $\mathrm{su}(2^N)$,

that is, the chain is fully controllable by modulating $\vec{B}_1(t)$ only!

How do we get the modulation $\vec{B}_1(t)$?

We still don't have an efficient method, thus need to rely on some numerics .

Krotov's method:

Fix t_i and t_f

Guess/change $\vec{B}_1(t)$

Repeat until $\varepsilon < \varepsilon_0$

Compute $\varepsilon = \left\| U_{\text{goal}} - \mathcal{T} \exp\left(-i \int_{t_i}^{t_f} H(t) dt \right) \right\|$

Classical computability of $\vec{B}_1(t)$

The computation time for Krotov $\propto$ Dimension of the space $\sim 2^N$

This difficulty can be circumvented when the inter-spin interaction is of the XX-type, such as

$$H^{\mathrm{int}} = \sum_n c_n [X_n X_{n+1} + Y_n Y_{n+1}]$$
$$= \sum_n c_n (a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n)$$

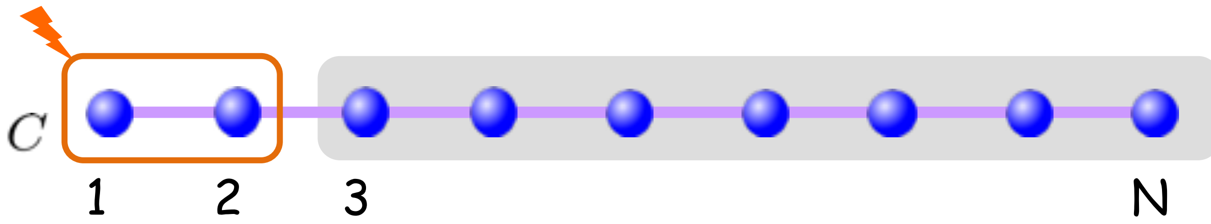
(after Jordan-Wigner transform)

which describes **non-interacting free fermions.**

The computational complexity can be reduced from $O(2^N)$ to $O(N)$.

Full control of a spin chain

With XX-type spin chain, any unitary is realisable by controlling the two end spins



Parameters to be modulated: magnetic fields at spins 1 and 2

no matter how complicated

Physical time to execute a unitary $\propto N^2$

theory to explain/predict this relation still missing... :-(

D. Burgarth, KM, M. Murphy, S. Montangero, T. Calarco, F. Nori, M.B. Plenio, PRA(R) (2010).

Quantum system identification under limited access

Can we get sufficient information on the system before controlling it?

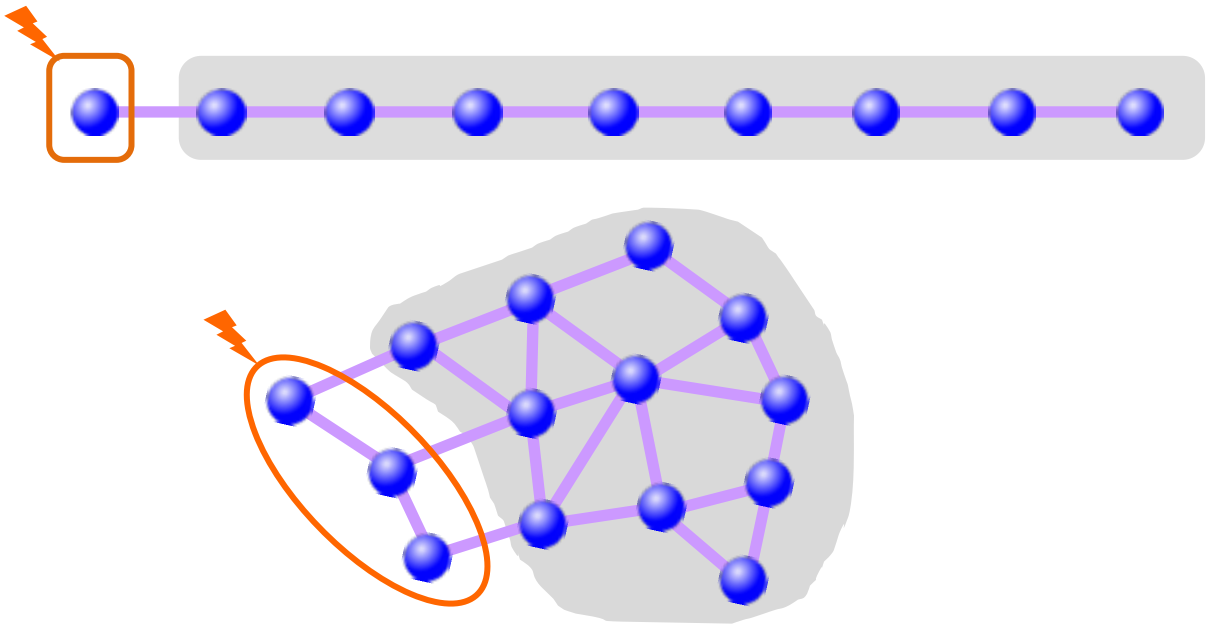
$=$ Can we know the Hamiltonian that governs the dynamics?

Most physical systems are more or less in this situation.

Yes, we can, but not always (naturally)

Quantum system identification under limited access

Spin networks: the entire system identifiable/controllable in many cases (including those of short coherence times),



provided a certain graph criterion is satisfied.

D. Burgarth, KM, F. Nori, PRA79, 020305(R) (2009).

D. Burgarth, KM, NJP11, 103019 (2009); NJP13, 013019(2011).

KM, D. Burgarth, A. Ishizaki, T. Takui, K.B. Whaley, QIC12, 736(2012).

Quantum system identification under limited access

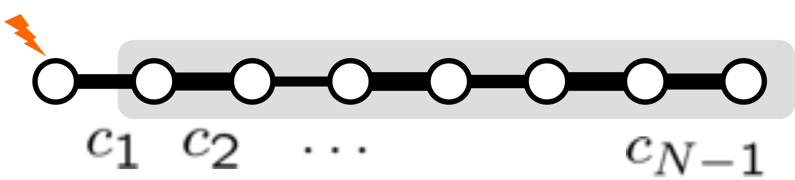
'excitation-preserving' Hamiltonians $[H, \sum Z_n] = 0$

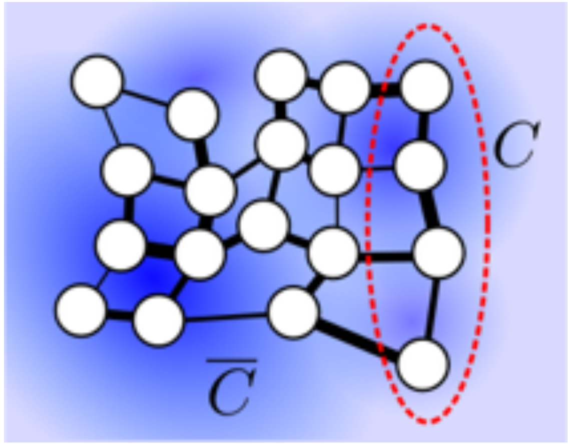
$$H = \sum_{(m,n)\in E} \boxed{c_{mn}}(X_m X_n + Y_m Y_n + \Delta Z_m Z_n) + \sum_{m\in V} \boxed{b_m} Z_m$$

Parameters to be estimated

$|\mathbf{n}\rangle \equiv |0...\underset{n}{1}...0\rangle$ stays in the single excitation subspace

Effectively, H describes hopping between sites.





2. Gateway scheme (1D spin chain)

Inject from the spin 1

The reduced density matrix for the spin 1

$$\frac{1}{\sqrt{2}}(|\mathbf{0}\rangle + |\mathbf{1}\rangle) \xrightarrow[\text{After a time lapse } t]{} \rho_1(t) = \frac{1}{2}\begin{pmatrix} 2-|f_{11}|^2 & e^{iGt/2}f_{11} \\ e^{-iGt/2}f_{11}^* & |f_{11}|^2 \end{pmatrix}$$

$$G = \Delta \sum_m c_m$$

$$f_{11} = \langle \mathbf{1}|e^{-iHt}|\mathbf{1}\rangle = \sum_j e^{-iE_jt}|\langle \mathbf{1}|E_j\rangle|^2$$

$\Rightarrow$ $\boxed{E_j,\ \langle E_j|\mathbf{1}\rangle \text{ obtained.}}$ $\langle E_j|\mathbf{1}\rangle$ can be taken as real, WLG.

Then, from $c_1\langle E_j|\mathbf{2}\rangle = \boxed{\langle E_j|H|\mathbf{1}\rangle} = E_j\langle E_j|\mathbf{1}\rangle$ and the normalisation $\sum_j |\langle E_j|\mathbf{2}\rangle|^2 = 1$, we get $|c_1|^2 = \sum_j E_j^2|\langle E_j|\mathbf{1}\rangle|^2$

$\Rightarrow$ $c_2\langle E_j|\mathbf{3}\rangle + c_1\langle E_j|\mathbf{1}\rangle = \boxed{\langle E_j|H|\mathbf{2}\rangle} = E_j\langle E_j|\mathbf{2}\rangle$

$c_2,\ \langle E_j|\mathbf{3}\rangle$ obtained. $\cdots$

PRA79, 020305(R) (2009)

2. Gateway scheme (generic spin networks)

- How about more general graphs, like, 2D, 3D, …?

  Possible to generalise the 1D gateway scheme.

  Need an extra graph-theoretic condition on the choice of accessible area, i.e., *'infection'*.

  NJP11, 103019 (2009)

- Excitation non-preserving hamiltonians?

  Quadratic hamiltonians:

$$H = \sum_{(m,n)\in E} A_{mn} a_m^\dagger a_n + \frac{1}{2}(B_{mn} a_m^\dagger a_n^\dagger + B_{mn}^* a_n a_m)$$

  which includes XX, Ising with a transverse field, etc.

  Gateway scheme can work for both **fermionic** and **bosonic** operators.

  NJP13, 013019 (2011)

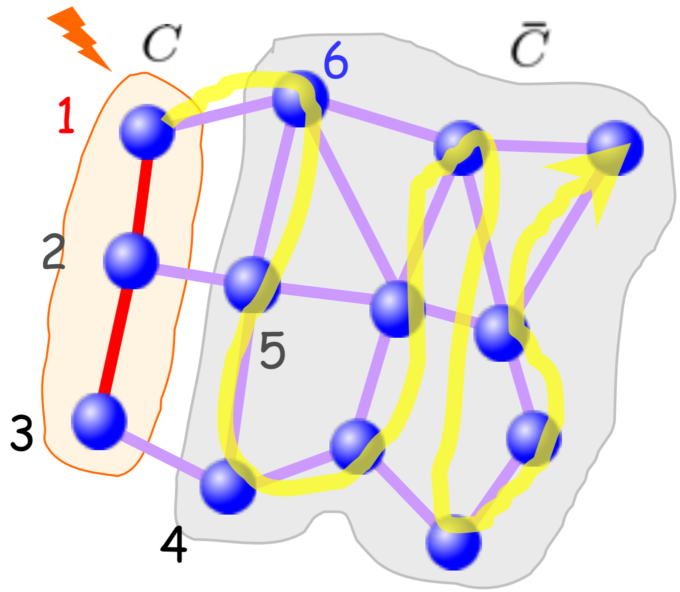
Graph infection

(i) Suppose nodes in C are 'infected' with some property.

(ii) If there is an infected node i that has a unique un-infected neighbour k, then k gets infected.

(iii) If eventually all nodes are infected, we say "C infects V".

$$V = C \cup \bar{C}$$

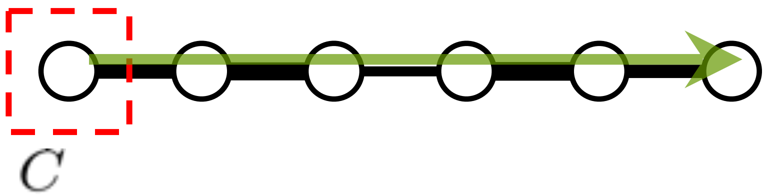


NJP11, 103019 (2009)

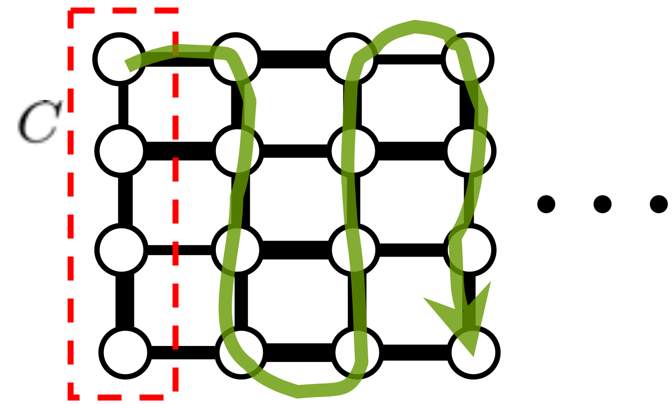
2. Example (generic spin networks)

Examples of infecting graphs :

1D chains



Square lattices



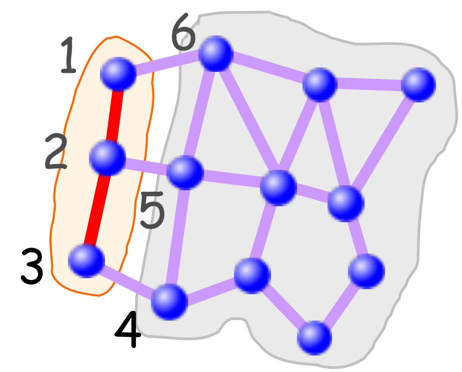
Interactions up to n-th nearest neighbours

NJP11, 103019 (2009)

2. Example (generic spin networks)

By state tomography, we obtain $f_{ii}(t)\ (i \in C)$, through which we acquire information on $\boxed{E_j \text{ and } \langle \mathbf{i} | E_j \rangle.}$ (if no degeneracy in the spectrum)

cf. $f_{ii}(t) = \langle \mathbf{i} | e^{-iHt} | \mathbf{i} \rangle = \sum_j e^{-iE_j t} |\langle \mathbf{i} | E_j \rangle|^2.$



To see how it works, assume $\Delta = 0,\ b_n = 0\ \forall n$.

(the scheme works when $\Delta \neq 0,\ b_n \neq 0$ as well.)

<u>Inside *C*</u>

$$c_{12} = \langle \mathbf{1} | H | \mathbf{2} \rangle = \sum \boxed{E_k} \boxed{\langle \mathbf{1} | E_k \rangle} \boxed{\langle E_k | \mathbf{2} \rangle}$$

$$c_{23} = \dots$$

obtained from tomography

2. Example (generic spin networks)

<u>Outside C</u>

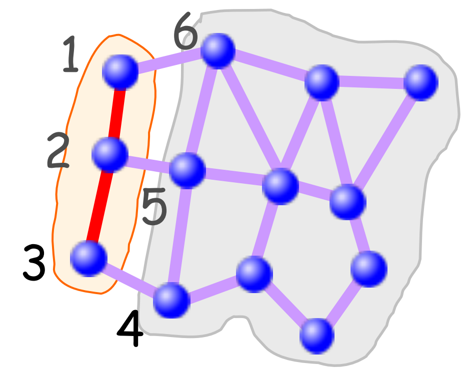
$$\langle \mathbf{1}|H|E_j\rangle = \boxed{c_{12}\langle \mathbf{2}|E_j\rangle} + \boxed{c_{16}}\boxed{\langle \mathbf{6}|E_j\rangle}$$

||

$$\boxed{E_j\langle \mathbf{1}|E_j\rangle}$$

Known

Not known



$\Rightarrow$ $c_{16}, \langle \mathbf{6}|E_j\rangle$ obtained. similarly, $c_{25}, c_{34}, \langle \mathbf{5}|E_j\rangle, \langle \mathbf{4}|E_j\rangle$.

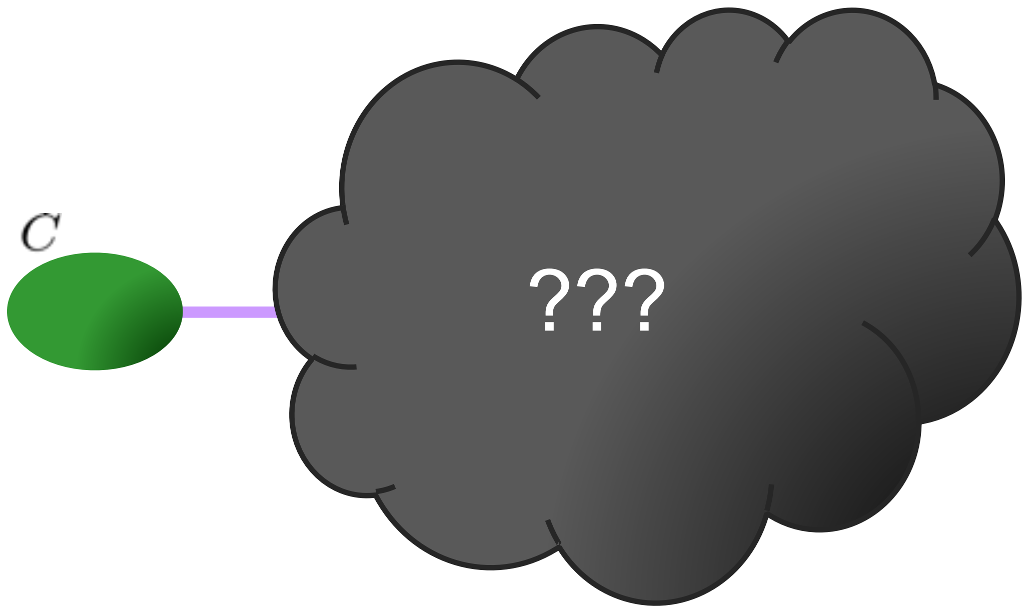
$\Rightarrow$ $c_{45} = \langle \mathbf{4}|H|\mathbf{5}\rangle = \sum E_k\langle \mathbf{4}|E_k\rangle\langle E_k|\mathbf{5}\rangle$

$c_{56}, c_{47}, \cdots$

The ‘graph infection’ property guarantees that there appears only one unknown term.

Quantum system identification under limited access

More general cases (very little a priori knowledge)



So far, we've obtained some positive results

- It's partially identifiable, and controllable
- Indistinguishable states form (dynamically) equivalence class

Still of mathematical interest, further progress needed

M. Owari, KM, T. Takui, G. Kato, PRA91, 012343 (2015)

Summary

The dynamical Lie algebra

Only a few modulable parameters can be sufficient to manipulate a large system

Spin systems

Controlling multi-spin systems with a limited number of control parameters is now becoming experimentally realisable.

Decoherence?

There could be a trick to utilise the environment in our favour.
Yesterday's enemies could be today's friends. ☺