Quantum systems exhibiting parameter-dependent spectral transitions

Pavel Exner

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague

in collaboration with Diana Barseghyan, Andrii Khrabustovskyi,

Vladimir Lotoreichik, and Miloš Tater

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Talk outline

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Smilansky model

The model was originally proposed in [Smilansky'04] to describe a one-dimensional system interacting with a caricature *heat bath* represented by a harmonic oscillator.

Mathematical properties of the model were analyzed in [Solomyak'04], [Evans-Solomyak'05], [Naboko-Solomyak'06]. More recently, time evolution in such a (slightly modified) model was analyzed [Guarneri'11]

In PDE terms, the model is described through a 2D Schrödinger operator

$$H_{\mathrm{Sm}} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)$$

on $L^2(\mathbb{R})$ with various modifications to be mentioned later.

Due to a particular choice of the coupling the model exhibited a *spectral transition* with respect to the coupling parameter λ.

A summary of results about the model

- *Spectral transition:* if $|\lambda| > \sqrt{2}$ the particle can escape to infinity along the singular 'channel' in the y direction. In spectral terms, it corresponds to switch from a positive to a below unbounded spectrum at $|\lambda| = \sqrt{2}$.
- At the heuristic level, the *mechanism* is easy to understand: we have an effective variable decoupling far from the x-axis and the oscillator potential competes there with the δ interaction eigenvalue $-\frac{1}{4}\lambda^2 y^2$.
- *Eigenvalue absence:* for any $\lambda \geq 0$ there are *no eigenvalues* $\geq \frac{1}{2}$. If $|\lambda| > \sqrt{2}$, the point spectrum of H_{Sm} is *empty*.
- *Existence of eigenvalues:* for $0 < |\lambda| < \sqrt{2}$ we have $H_{\mathrm{Sm}} \geq 0$. The point spectrum is nonempty and finite, and

$$N(\tfrac{1}{2}, H_{\mathrm{Sm}}) \sim \frac{1}{4\sqrt{2(\mu(\lambda)-1)}}$$

holds as $\lambda \to \sqrt{2}-$, where $\mu(\lambda) := \sqrt{2}/\lambda$.

Further results

- *Absolute continuity:* in the supercritical case $|\lambda| > \sqrt{2}$ we have $\sigma_{\mathrm{ac}}(H_{\mathrm{Sm}}) = \mathbb{R}$

- *Extension* of the result to a two 'channel' case with different oscillator frequencies [Evans-Solomyak'05]

- *Extension* to multiple 'channels' on a system *periodic in x* [Guarneri'11]. In this paper the time evolution generated by H_{Sm} is investigated and proposed as a model of *wavepacket collapse*.

- The above results have been obtained by a combination of different methods: a reduction to an infinite system of ODE's, facts from Jacobi matrices theory, variational estimates, etc.

Before proceeding further, let show how the spectrum can be *treated numerically* in the subcritical case.

Numerical search for eigenvalues

In the halfplanes $\pm x > 0$ the wave functions can be expanded using the 'transverse' base spanned by the functions

$$\psi_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \, \mathrm{e}^{-y^2/2} H_n(y)$$

corresponding to the oscillator eigenvalues $n + \frac{1}{2},\ n = 0, 1, 2, \dots$.

Furthermore, one can make use of the mirror symmetry w.r.t. $x = 0$ and divide H_λ into the trivial odd part $H_\lambda^{(-)}$ and the even part $H_\lambda^{(+)}$ which is equivalent to the operator on $L^2(\mathbb{R} \times (0, \infty))$ with the same symbol determined by the boundary condition

$$f_x(0+, y) = \frac{1}{2} \alpha y f(0+, y) \, .$$

Numerical solution, continued

We substitute the Ansatz

$$f(x,y)=\sum_{n=0}^{\infty} c_n\,\mathrm{e}^{-\kappa_n x}\psi_n(y)$$

with $\kappa_n := \sqrt{n+\frac{1}{2}-\epsilon}$.

This yields for solution with the energy ϵ the equation

$$B_\lambda c = 0\,,$$

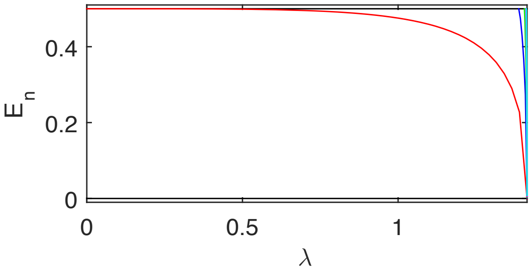
where c is the coefficient vector and B_λ is the operator in ℓ^2 with

$$(B_\lambda)_{m,n} = \kappa_n\delta_{m,n} + \frac{1}{2}\lambda(\psi_m, y\psi_n)\,.$$

Note that the matrix is in fact tridiagonal because

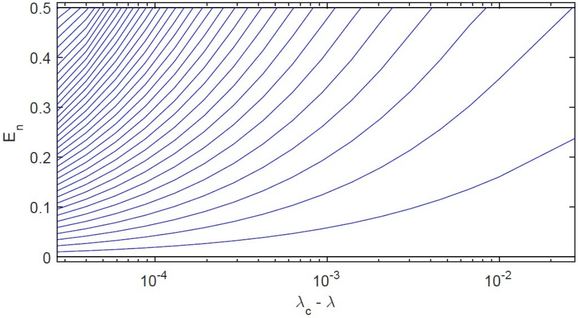
$$(\psi_m, y\psi_n) = \frac{1}{\sqrt{2}}\big(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\big)\,.$$

Smilansky model eigenvalues



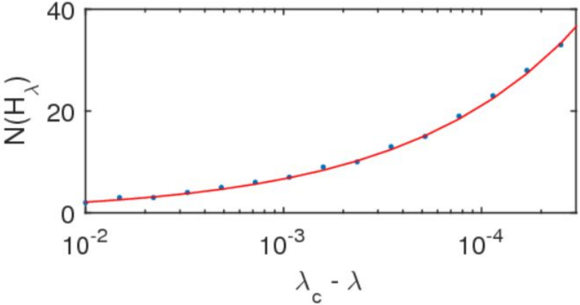
In most part of the subcritical region there is a single eigenvalue, the second one appears only at $\lambda \approx 1.387559$. The next thresholds are $1.405798, 1.410138, 1.41181626, 1.41263669, \ldots$

Smilansky model eigenvalues



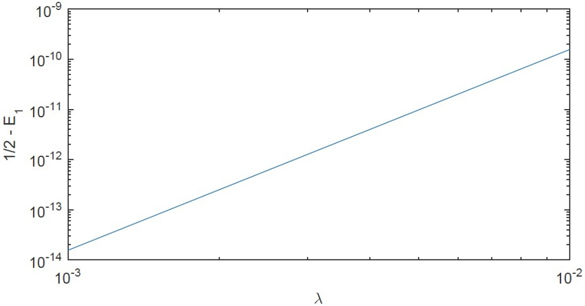
Close to the critical value, however, many eigenvalues appear which gradually fill the interval $(0, \frac{1}{2})$ as the critical value is approached

Their number is as predicted



The dots mean the eigenvalue numbers, the red curve is the above mentioned asymptotics due to Solomyak

Smilansky model ground state



The numerical solution also indicates other properties, for instance, that the first eigenvalue behaves as $\epsilon_1 = \frac{1}{2} - c\lambda^4 + o(\lambda^4)$ as $\lambda \to 0$, with $c \approx 0.0156$.

In fact, we have $c = 0.015625$

Indeed, the relation $B_\lambda c = 0$ can be written explicitly as

$$\sqrt{\mu_\lambda}\, c_0^\lambda + \frac{\lambda}{2\sqrt{2}} c_1^\lambda = 0\,,$$

$$\frac{\sqrt{k}\lambda}{2\sqrt{2}} c_{k-1}^\lambda + \sqrt{k+\mu_\lambda}\, c_k^\lambda + \frac{\sqrt{k+1}\lambda}{2\sqrt{2}} c_{k+1}^\lambda = 0\,, \qquad k \ge 1\,,$$

where $\mu_\lambda := \frac{1}{2} - E_1(\lambda)$ and $c^\lambda = \{c_0^\lambda, c_1^\lambda, \dots\}$ is the corresponding normalized eigenvector of B_λ.

Using the above relations and simple estimates, we get

$$\sum_{k=1}^{\infty} |c_k^\lambda|^2 \le \frac{3}{4}\lambda^2 \quad \text{and} \quad c_0^\lambda = 1 + \mathcal{O}(\lambda^2)$$

as $\lambda \to 0+$; hence we have in particular $c_1^\lambda = \frac{\lambda}{2\sqrt{2}} + \mathcal{O}(\lambda^2)$.

In fact, we have $c = 0.015625$

The first of the above relation then gives

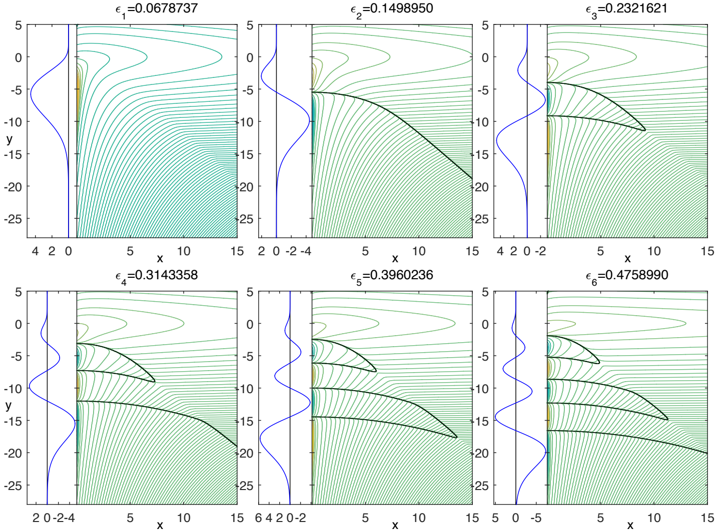
$$\mu_\lambda = \frac{\lambda^4}{64} + \mathcal{O}(\lambda^5)$$

as $\lambda \to 0+$, in other words

$$E_1(\lambda) = \frac{1}{2} - \frac{\lambda^4}{64} + \mathcal{O}(\lambda^5)\,.$$

And the mentioned coefficient 0.015625 is nothing else than $\frac{1}{64}$. $\square$.

Smilansky model eigenfunctions



The first six eigenfunctions of $H_{\rm Sm}$ for $\lambda = 1.4128241$, in other words, $\lambda = \sqrt{2} - 0.0086105$.

A regular version of Smilansky model

Our next aim is to show that one can observe a similar effect for Schrödinger operators with regular potentials. Now, however, the coupling cannot be linear in y and the profile of the channel has to change with y.

Recall that the effect comes from competition between the oscillator potential with the principal eigenvalue of the 'transverse' part of the operator equal to $\frac{1}{4}\lambda^2 y^2$.

We replace the δ by a family of shrinking potentials whose mean matches the δ coupling constant, $\int U(x,y)\,\mathrm{d}x \sim y$. This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function V.

This motivates us to investigate the following operator on $L^2(\mathbb{R}^2)$,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy)\chi_{\{|x|\le a\}}(x),$$

where ω, a are positive constants, $\chi_{\{|y|\le a\}}$ is the indicator function of the interval $(-a,a)$, and the potential V with $\operatorname{supp} V \subset [-a,a]$ is a nonnegative function with bounded first derivative.

A regular version of Smilansky model, continued

By Faris-Lavine theorem the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ and the same is true for its generalization,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \sum_{j=1}^{N} \lambda_j y^2 V_j(xy) \chi_{\{|x-b_j| \le a_j\}}(x)$$

with a finite number of channels, where functions V_j are positive with bounded first derivative, with the supports contained in $(b_j - a_j, b_j + a_j)$ and such that $\mathrm{supp}\, V_j \cap \mathrm{supp}\, V_k = \emptyset$ holds for $j \neq k$.

Remark

We note that the properties discussed below depend on the asymptotic behavior of the potential channels and would not change if the potential is modified in the vicinity of the x-axis, for instance, by replacing the above cut-off functions with $\chi_{|y| \ge a}(y)$ and $\chi_{|y| \ge a_j}(y)$, respectively.

Subcritical case

To state the result we employ a 1D comparison operator $L = L_V$,

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 - \lambda V(x)$$

on $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$. What matters is the sign of its spectral threshold; since V is supposed to be nonnegative, the latter is a monotonous function of λ and there is a $\lambda_{\mathrm{crit}} > 0$ at which the sign changes.

Theorem (Barseghyan-E'14)

Under the stated assumption, the spectrum of the operator H is bounded from below provided the operator L is positive.

Proof outline

It is sufficient to prove the claim for $\lambda = 1$. We employ Neumann bracketing, similarly as for the previous model.

Let h_n and $\widetilde{h}_n$ be respectively the restrictions of operator H to the strips $G_n = \mathbb{R} \times \{y:\, \ln n < y \le \ln(n+1)\}$, $n = 1, 2, \dots$, and $\widetilde{G}_n$, their mirror images w.r.t. y, with Neumann boundary conditions; then

$$H \ge \bigoplus_{n=1}^{\infty} (h_n \oplus \widetilde{h}_n)\,;$$

We find a uniform lower bound $\sigma(h_n)$ and $\sigma(\widetilde{h}_n)$ as $n \to \infty$.

Using the assumptions about V we find

$$V(xy) - V(x \ln n) = \mathcal{O}\left(\frac{1}{n \ln n}\right), \quad y^2 - \ln^2 n = \mathcal{O}\left(\frac{\ln n}{n}\right)$$

for any $(x, y) \in G_n$, and analogous relations for $\widetilde{G}_n$.

Proof outline – continued

This yields

$$y^2 V(xy) - \ln^2 n\, V(x \ln n) = \mathcal{O}\left(\frac{\ln n}{n}\right)$$

for for any $(x, y) \in \widetilde{G}_n$ which allows us to check that

$$\inf \sigma(h_n) \geq \inf \sigma(l_n) + \mathcal{O}\left(\frac{\ln n}{n}\right),$$

where $l_n := -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 \ln^2 n - \ln^2 n\, V(x \ln n)$ on $L^2(G_n)$.

The analogous relation holds for $\widetilde{l}_n$ on $L^2(\widetilde{G}_n)$. It is important that all these operators have separated variables.

Since the minimal eigenvalue of Neumann Laplacian $-\frac{\mathrm{d}^2}{\mathrm{d}y^2}$ on the strips $\ln n < y \leq \ln(n+1)$, $n = 1, 2, \dots,$ is zero, we have $\inf \sigma(l_n) = \inf \sigma\big(l_n^{(1)}\big)$, where the last operator on $L^2(\mathbb{R})$ acts as

$$l_n^{(1)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 \ln^2 n - \ln^2 n\, V(x \ln n)$$

Proof outline – concluded

Note that the cut-off function $\chi_{\{|x|\le a\}}$ plays no role in the asymptotic estimate as it affects a finite number of terms only.

By the change of variable $x = \frac{t}{\ln n}$ the last operator is unitarily equivalent to $\ln^2 n\, L$ which is non-negative as long as L is non-negative. In the same way one proves that $\widetilde{l}_n$ is non-negative ; this concludes the proof. $\square$

In the same way one can treat systems restricted in the x direction:

Corollary

Let H be 'our' operator on $(-c,c)\times\mathbb{R}$ for some $c \ge a$ with Dirichlet (Neumann, periodic) boundary conditions in the variable x. The spectrum of H is bounded from below if $L \ge 0$ holds, where L is the comparison operator on $L^2(-c,c)$ with Dirichlet (respectively, Neumann or periodic) boundary conditions.

Supercritical case

Once the transverse channel principal eigenvalue dominates over the harmonic oscillator contribution, the spectral behavior changes:

Theorem (Barseghyan-E'14)

Under our hypotheses, $\sigma(H) = \mathbb{R}$ *holds if* $\inf \sigma(L) < 0$.

Proof relies on construction of an appropriate Weyl sequence: we have to find $\{\psi_k\}_{k=1}^{\infty} \subset D(H)$ such that $\|\psi_k\| = 1$ which contains no convergent subsequence, and at same time

$$\|H\psi_k - \mu\psi_k\| \to 0 \quad \text{as} \quad k \to \infty.$$

The construction is rather technical and we sketch just the main steps.

Proof outline

The claim is invariant under scaling transformations, hence we may suppose $\inf \sigma(L) = -1$. The spectral threshold is a simple isolated eigenvalue; we denote the corresponding normalized eigenfunction by h.

We want to show first that $0 \in \sigma_{\mathrm{ess}}(H)$. In fact, it would be enough for the proof to show that zero belongs to $\sigma(H)$ but we get the stronger claim at no extra expense.

We fix an $\varepsilon > 0$ and choose a natural $k = k(\varepsilon)$ with which we associate a function $\chi_k \subset C_0^2(1,k)$ satisfying the following conditions

$$\int_1^k \frac{1}{z}\chi_k^2(z)\,\mathrm{d}z = 1 \quad \text{and} \quad \int_1^k z(\chi_k'(z))^2\,\mathrm{d}z < \varepsilon.$$

Proof outline – continued

Such functions exist: as an example consider

$$\tilde{\chi}_k(z) = \frac{8\ln^3 z}{\ln^3 k}\chi_{\{1\le z\le\sqrt{k}\}}(z) + \frac{2\ln k - 2\ln z}{\ln k}\chi_{\{\sqrt{k}+1\le z\le k-1\}}(z)$$
$$+g_k(z)\chi_{\{\sqrt{k}<z<\sqrt{k}+1\}}(z) + q_k(z)\chi_{\{k-1<z\le k\}}(z),$$

where g_k and q_k are interpolating functions chosen in such a way that $\tilde{\chi}_k \in C_0^2(1,k)$, and define

$$\chi_k(z) = \left(\int_1^k \frac{1}{z}\tilde{\chi}_k^2(z)\,\mathrm{d}z\right)^{-1/2} \tilde{\chi}_k(z).$$

Given such functions χ_k, put

$$\psi_k(x,y) := h(xy)\,\mathrm{e}^{iy^2/2}\chi_k\left(\frac{y}{n_k}\right) + \frac{f(xy)}{y^2}\,\mathrm{e}^{iy^2/2}\chi_k\left(\frac{y}{n_k}\right),$$

where $f(t) := -\frac{i}{2}\,t^2 h(t),\ t\in\mathbb{R}$, and $n_k\in\mathbb{N}$ is a positive integer, which we choose using the following auxiliary result.

Proof outline – continued

Lemma

Let ψ_k, $k = 1, 2, \dots$, as defined above; then for any given k one can achieve that $\|\psi_k\|_{L^2(\mathbb{R}^2)} \geq \frac{1}{2}$ holds by choosing n_k large enough.

We need one more auxiliary result:

Lemma

Let ψ_k, $k = 1, 2, \dots$, be again functions defined above; then the inequality $\|H\psi_k\|^2_{L^2(\mathbb{R}^2)} < c\varepsilon$ with a fixed constant c holds for $k = k(\varepsilon)$.

Proofs are in both cases straightforward but rather tedious.

Proof outline – concluded

Using the lemmata, we are able to complete the proof. We fix a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $\varepsilon_j \searrow 0$ holds as $j \to \infty$ and to any j we construct a function $\psi_{k(\varepsilon_j)}$ in such a way that $n_{k(\varepsilon_j)} > k(\varepsilon_{j-1}) n_{k(\varepsilon_{j-1})}$.

The norms of $H\psi_{k(\varepsilon_j)}$ are bounded from above with $9\varepsilon_j$ on the right-hand side, and since the supports of $\psi_{k(\varepsilon_j)}$, $j = 1, 2, \dots,$ do not intersect each other by construction, their sequence converges weakly to zero.

This yields the sought Weyl sequence for zero energy; for any nonzero real number μ we use the same procedure replacing the above ψ_k with

$$\psi_k(x,y) = h(xy)\, \mathrm{e}^{i\epsilon_\mu(y)} \chi_k \left(\frac{y}{n_k}\right) + \frac{f(xy)}{y^2}\, \mathrm{e}^{i\epsilon_\mu(y)} \chi_k \left(\frac{y}{n_k}\right),$$

where $\epsilon_\mu(y) := \int_{\sqrt{|\mu|}}^{y} \sqrt{t^2 + \mu}\, \mathrm{d}t$, and furthermore, the functions f, χ_k are defined in the same way as above. $\square$

Restricted motion

In the supercritical case, too, the result extends to systems restricted in the x direction:

Theorem

Let H be the 'our' operator on $L^2(-c,c) \otimes L^2(\mathbb{R})$ for some $c > 0$ with Dirichlet condition at $x = \pm c$ and denote by L the corresponding Dirichlet operator on $L^2(-c,c)$. If the spectral threshold of L is negative, the spectrum of H covers the whole real axis.

Observing the domains of the quadratic form associated with such operators we get

Corollary

The claim of the above theorem remains valid if the Dirichlet boundary conditions at $x = \pm c$ are replaced by any other self-adjoint boundary conditions.

The multichannel case

The above results are interesting not only *per se* or to deal with the Guarneri-type periodic modification of the model.

Using a simple bracketing argument we can show how the spectral-regime transition looks like in the multichannel case:

Theorem (Barseghyan-E'14)

Let H be 'our' operator with the potentials satisfying the stated assumptions, namely the functions V_j are positive with bounded first derivative and $\mathrm{supp}\, V_j \cap \mathrm{supp}\, V_k = \emptyset$ holds for $j \neq k$. Denote by L_j the comparison operator on $L^2(\mathbb{R})$ with the potential V_j and set $t_V := \min_j \inf \sigma(L_j)$. Then H is bounded from below if and only if $t_V \geq 0$ and in the opposite case its spectrum covers the whole real axis.

Another model class

Consider next a related family of systems in which the transition is even *more dramatic* passing from *purely discrete spectrum* in the subcritical case to the whole real line in the supercritical one.

Recall that there are situations where *Weyl's law fails* and the spectrum is discrete even if the classically allowed phase-space volume is infinite. A classical example due to [Simon'83] is a 2D Schrödinger operator with the potential

$$V(x,y) = x^2y^2$$

or more generally, $V(x,y) = |xy|^p$ with $p \geq 1$.

Similar behavior one can observe for Dirichlet Laplacians in *regions with hyperbolic cusps* – see [Geisinger-Weidl'11] for recent results and a survey. Moreover, using the *dimensional-reduction technique* of Laptev and Weidl one can prove spectral estimates for such operators.

A common feature of these models is that the particle motion is confined into *channels narrowing towards infinity*.

Adding potentials unbounded from below

This may remain true even for Schrödinger operators with *unbounded from below* in which a classical particle can escape to infinity with an increasing velocity.

The situation changes, however, if the *attraction is strong enough*

As an illustration, let us analyze the following class of operators:

$$L_p(\lambda) \,:\; L_p(\lambda)\psi = -\Delta\psi + \left(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\right)\psi\,, \quad p \ge 1$$

on $L^2(\mathbb{R}^2)$, where (x,y) are the standard Cartesian coordinates in $\mathbb{R}^2$ and the parameter λ in the second term of the potential is non-negative; unless the value of λ is important we write it simply as L_p.

Note that $\frac{2p}{p+2} < 2$ so the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ by Faris-Lavine theorem again; the symbol L_p or $L_p(\lambda)$ will always mean its closure.

The subcritical case

The spectral properties of $L_p(\lambda)$ depend crucially on the value of λ and there is a transition between different regimes as λ changes.

Let us start with the subcritical case which occurs for small values of λ. To characterize the smallness quantitatively we need an auxiliary operator which will be an (an)harmonic oscillator Hamiltonian on line,

$$\tilde{H}_p \,:\, \tilde{H}_p u = -u'' + |t|^p u$$

on $L^2(\mathbb{R})$ with the standard domain. Let γ_p be the minimal eigenvalue of this operator; in view of the potential symmetry we have $\gamma_p = \inf \sigma(H_p)$, where

$$H_p \,:\, H_p u = -u'' + t^p u$$

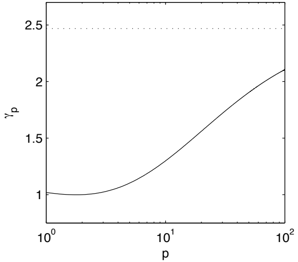
on $L^2(\mathbb{R}_+)$ with Neumann condition at $t = 0$.

The subcritical case – continued

The eigenvalue $\gamma_p = \inf \sigma(H_p)$ equals one for $p = 2$; for $p \to \infty$ it becomes $\gamma_\infty = \frac{1}{4}\pi^2$; it smoothly interpolates between the two values.

Since $x^p \geq 1 - \chi_{[0,1]}(x)$ we have $\gamma_p \geq \epsilon_0 \approx 0.546$, where ϵ_0 is the ground-state energy of the rectangular potential well of depth one.

In fact, a numerical solution gives true minimum $\gamma_p \approx 0.998995$ attained at $p \approx 1.788$; in the semilogarithmic scale the plot is as follows:



The subcritical case – continued

The spectrum is naturally bounded from below and discrete if $\lambda = 0$; our aim is to show that this remains to be the case provided λ is small enough.

Theorem (E-Barseghyan'12)

For any $\lambda \in [0, \lambda_{\mathrm{crit}}]$, where $\lambda_{\mathrm{crit}} := \gamma_p$, the operator $L_p(\lambda)$ is bounded from below for $p \geq 1$; if $\lambda < \gamma_p$ its spectrum is purely discrete.

Idea of the proof: Let $\lambda < \gamma_p$. By minimax we need to estimate L_p from below by a s-a operator with a purely discrete spectrum. To construct it we employ bracketing imposing additional Neumann conditions at concentric circles of radii $n = 1, 2, \dots$.

In the estimating operators the variables decouple asymptotically and the spectral behavior is determined by the angular part of the operators.

Subcritical behavior – the proof

Specifically, in polar coordinates we get direct sum of operators acting as

$$L^{(1)}_{n,p}\psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{1}{n^2}\frac{\partial^2\psi}{\partial\varphi^2} + \left(\frac{r^{2p}}{2^p}|\sin 2\varphi|^p - \lambda r^{2p/(p+2)}\right)\psi$$

on the annuli $G_n := \{(r,\varphi)\,:\; n-1 \le r < n,\; 0 \le \varphi < 2\pi\}$, $n = 1, 2, \dots$ with Neumann conditions imposed on ∂G_n.

Obviously $\sigma(L^{(1)}_{n,p})$ is purely discrete for each $n = 1, 2, \dots$, hence it is sufficient to check that $\inf\sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$.

We estimate $L^{(1)}_{n,p}$ from below by an operator with separating variables, note that the radial part does not contribute and use the symmetry of the problem; for $\varepsilon \in (0,1)$ the question is then to analyze

$$L^{(2)}_{n,p}\,:\; L^{(2)}_{n,p}u = -u'' + \left(\frac{n^{2p+2}}{2^p}\sin^p 2x - \frac{\lambda}{1-\varepsilon}\,n^{(4p+4)/(p+2)}\right)u$$

on $L^2(0,\pi/4)$ with Neumann conditions, $u'(0) = u'(\pi/4) = 0$.

Subcritical behavior – proof continued

We have $n^2 \inf \sigma(L^{(1)}_{n,p}) \geq \inf \sigma(L^{(2)}_{n-1,p})$ if n is large enough, specifically for $n > \left(1-(1-\varepsilon)^{(p+2)/(4p+4)}\right)^{-1}$, hence it is sufficient to investigate the spectral threshold $\mu_{n,p}$ of $L^{(2)}_{n,p}$ as $n \to \infty$.

The trigonometric potential can be estimated by a powerlike one with the similar behavior around the minimum introducing, e.g.

$$L^{(3)}_{n,p} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + n^{2p+2}x^p \left(\chi_{(0,\delta(\varepsilon)]}(x) + \left(\frac{2}{\pi}\right)^p \chi_{[\delta(\varepsilon),\pi/4)}(x)\right) - \lambda'_\varepsilon\, n^{(4p+4)/(p+2)}$$

for small enough $\delta(\varepsilon)$ with Neumann boundary conditions at $x = 0, \frac{1}{4}\pi$, where we have denoted $\lambda'_\varepsilon := \lambda(1-\varepsilon)^{-p-1}$.

We have $L^{(2)}_{n,p} \geq (1-\varepsilon)^p L^{(3)}_{n,p}$. To estimate the *rhs* by comparing the indicated potential contributions it is useful to pass to the rescaled variable $x = t \cdot n^{-(2p+2)/(p+2)}$.

Subcritical behavior – proof concluded

In this way we find that $\mu'_{n,p} := \inf \sigma(L^{(3)}_{n,p})$ satisfies

$$\frac{\mu'_{n,p}}{n^2} \to \infty \qquad \text{as} \qquad n \to \infty$$

Through the chain of inequalities we come to conclusion that $\inf \sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$ which proves discreteness of the spectrum for $\lambda < \gamma_p$.

If $\lambda = \gamma_p$ the sequence of spectral thresholds no longer diverges but it remains bounded from below and the same is by minimax principle true for the operator $L_p(\lambda)$. $\square$

Remark

It is natural to conjecture that $\sigma(L_p(\gamma_p)) \supset \mathbb{R}_+$. There may be a negative discrete spectrum in the critical case; we return to this question a little later.

The supercritical case

Theorem (E-Barseghyan'12)

The spectrum of $L_p(\lambda)$, $p \ge 1$, is unbounded below from if $\lambda > \lambda_{\mathrm{crit}}$.

Idea of the proof: Similar as above with a few differences:

- now we seek an upper bound to $L_p(\lambda)$ by a below unbounded operator, hence we impose Dirichlet conditions on concentric circles
- the estimating operators have now a nonzero contribution from the radial part, however, it is bounded by π^2 independently of n
- the negative λ-dependent term now outweights the anharmonic oscillator part so that $\inf \sigma(L_{n,p}^{(1,D)}) \to -\infty$ holds as $n \to \infty$ $\square$

Using suitable Weyl sequences similar to those the previous model, however, we are able to get a stronger result:

Theorem (Barseghyan-E'15)

$\sigma(L_p(\lambda)) = \mathbb{R}$ *holds for any* $\lambda > \gamma_p$ *and* $p > 1$.

Spectral estimates: bounds to eigenvalue sums

Let us return to the subcritical case and define the following quantity:

$$\alpha := \frac{1}{2}\left(1+\sqrt{5}\right)^2 \approx 5.236 > \gamma_p^{-1}$$

We denote by $\{\lambda_{j,p}\}_{j=1}^{\infty}$ the eigenvalues of $L_p(\lambda)$ arranged in the ascending order; then we can make the following claim.

Theorem (E-Barseghyan'12)

To any nonnegative $\lambda < \alpha^{-1} \approx 0.19$ there exists a positive constant C_p depending on p only such that the following estimate is valid,

$$\sum_{j=1}^{N} \lambda_{j,p} \geq C_p(1-\alpha\lambda)\frac{N^{(2p+1)/(p+1)}}{(\ln^p N+1)^{1/(p+1)}} - c\lambda N, \quad N = 1, 2, \dots,$$

where $c = 2\left(\frac{\alpha^2}{4}+1\right) \approx 15.7$.

Cusp-shaped regions

The above bounds are valid for any $p \geq 1$, hence it is natural to ask about the limit $p \to \infty$ describing the particle confined in a region with four hyperbolic 'horns', $D = \{(x, y) \in \mathbb{R}^2 : \, |xy| \leq 1\}$, described by the Schrödinger operator

$$H_D(\lambda) : \; H_D(\lambda)\psi = -\Delta\psi - \lambda(x^2 + y^2)\psi$$

with a parameter $\lambda \geq 0$ and Dirichlet condition on the boundary ∂D.

Theorem (E-Barseghyan'12)

The spectrum of $H_D(\lambda)$ is discrete for any $\lambda \in [0, 1)$ and the spectral estimate

$$\sum_{j=1}^{N} \lambda_j \geq \; C(1 - \lambda)\frac{N^2}{1 + \ln\, N}\,, \qquad N = 1, 2, \dots$$

holds true with a positive constant C.

Proof outline

To get the estimate for cusp-shaped regions, one can check that for any $u \in H^1$ satisfying the condition $u|_{\partial D} = 0$ the inequality

$$\int_D (x^2 + y^2) u^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \le \int_D |(\nabla u)(x,y)|^2\, \mathrm{d}x\, \mathrm{d}y$$

is valid which in turn implies

$$H_D(\lambda) \ge -(1-\lambda)\Delta_D\,,$$

where Δ_D is the Dirichlet Laplacian on the region D.

The result then follows from the eigenvalue estimates on Δ_D known from [Simon'83], [Jakšić-Molchanov-Simon'92].

The proof for $p \in (1, \infty)$ is more complicated, using splitting of $\mathbb{R}^2$ into rectangular domains and estimating contributions from the channel regions, the middle part, and the rest. We will not discuss it here, because we are able to demonstrate a stronger result *à la* Lieb and Thirring.

Better spectral estimates

Theorem (Barseghyan-E'15)

Given $\lambda < \gamma_p$, let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $L_p(\lambda)$. Then for $\Lambda \ge 0$ and $\sigma \ge 3/2$ the following inequality is valid,

$$\operatorname{tr}(\Lambda - L_p(\lambda))_+^\sigma \le C_{p,\sigma}\left(\frac{\Lambda^{\sigma+(p+1)/p}}{(\gamma_p-\lambda)^{\sigma+(p+1)/p}}\ln\left(\frac{\Lambda}{\gamma_p-\lambda}\right) + C_\lambda^2\left(\Lambda + C_\lambda^{2p/(p+2)}\right)^{\sigma+1}\right),$$

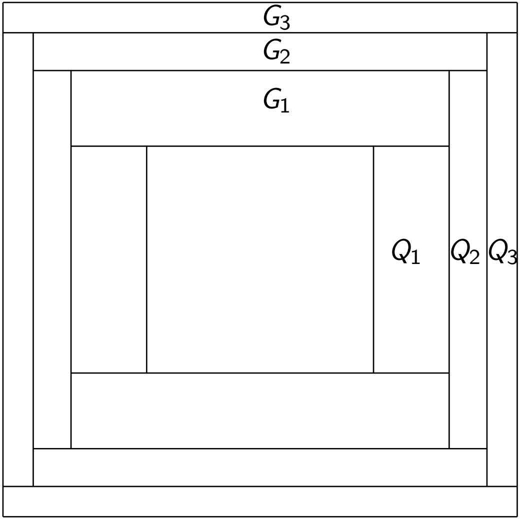
where the constant $C_{p,\sigma}$ depends on p and σ only and

$$C_\lambda =: \max\left\{\frac{1}{(\gamma_p-\lambda)^{(p+2)/(p(p+1))}}, \frac{1}{(\gamma_p-\lambda)^{(p+2)^2/(4p(p+1))}}\right\}.$$

Sketch of the proof: By minimax principle we can estimate $L_p(\lambda)$ from below by a self-adjoint operator with a purely discrete negative spectrum and derive a bound to the momenta of the latter.

We split $\mathbb{R}^2$ again, now in a 'lego' fashion using a monotone sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \to \infty$ and $\alpha_{n+1} - \alpha_n \to 0$ holds as $n \to \infty$.

Proof sketch



$x = \quad \alpha_1 \quad \alpha_2\ \alpha_3 \ldots$

Proof sketch, continued

Estimating the 'transverse' variables by by their extremal values, we reduce the problem essentially to assessment of the spectral threshold of the anharmonic oscillator with Neumann cuts.

Lemma

Let $l_{k,p} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^p$ *be the Neumann operator on* $[-k, k]$, $k > 0$. *Then*

$$\inf \sigma\left(l_{k,p}\right) \geq \gamma_p + o\left(k^{-p/2}\right) \quad \text{as } k \to \infty.$$

Combining it with the 'transverse' eigenvalues $\left\{ \frac{\pi^2 k^2}{(\alpha_{n+1}-\alpha_n)^2} \right\}_{k=0}^{\infty}$, using Lieb-Thirring inequality for this situation [Mickelin'15], and choosing properly the sequence $\{\alpha_n\}_{n=1}^{\infty}$, we are able to prove the claim. $\square$

The critical case

Let us return to $L := -\Delta + |xy|^p - \gamma_p(x^2+y^2)^{p/(p+2)}$ and the conjectures we made about its spectrum. Concerning the essential spectrum:

Theorem (Barseghyan-E-Khrabustovskyi-Tater'15)

We have $\sigma_{\mathrm{ess}}(L) \supset [0,\infty)$.

This can be proved in the same as above using suitable Weyl sequences.

Theorem (Barseghyan-E-Khrabustovskyi-Tater'15)

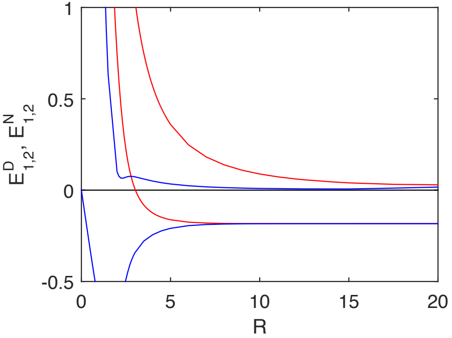
The negative spectrum of L is discrete.

The proof uses a 'lego' estimate similar to the one presented above.

For the moment, however, we cannot prove that $\sigma_{\mathrm{disc}}(L)$ is nonempty. We conjecture that it is the case having a *strong numerical evidence* for that.

Bracketing: numerical analysis

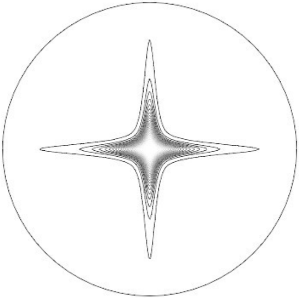
We solve our spectral problem with $p = 2$ in a disc of radius R with Dirichlet and Neumann condition at the boundary, and plot the first two eigenvalues as a function of R.



This indicates that the original critical problem has for $p = 2$ a single eigenvalue $E_1 \approx -0.18365$.

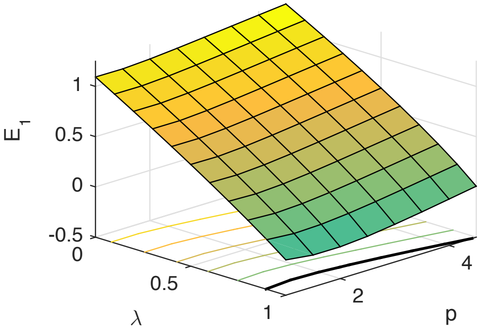
Ground state eigenfunction

We also find the eigenfunction, note that with the $R = 20$ cut-off the Dirichlet and Neumann ones are practically identical; the outer level marks the 10^{-3} value.



The dependence on p

Plotting the lowest eigenvalue in the subcritical case, we see that for small enough p there is likely a critical value $\lambda(p)$ at which $L_p(\lambda)$ loses positivity. On the other hand, it seems that for $p \gtrsim 20.392$ the critical operator might be positive



Summary & open questions

- We have analyzed spectral transitions in several classes of model coming from competition between a below positive and negative contributions of energy appearing in such potential 'channels'
- Various questions remain open, for instance, about the properties of the $\sigma_{\mathrm{disc}}(L)$ indicated numerically
- More generally, if the potential channels are regular and one has more than one transverse eigenvalue, one can conjecture that the spectral multiplicity will become larger after crossing each such threshold
- One can also conjecture that the spectrum will be absolutely continuous on the supercritical case, as it is established for the original Smilansky model
- In the case on Smilansky model when there is a continuous spectrum in the subcritical case one can expect the existence of resonances, etc.

The talk was based on

[EB12] P.E., D. Barseghyan: Spectral estimates for a class of Schrödinger operators with infinite phase space and potential unbounded from below, *J. Phys. A: Math. Theor.* **A45** (2012), 075204.

[BE14] D. Barseghyan, P.E.: A regular version of Smilansky model, *J. Math. Phys.* **55** (2014), 042194.

[ELT15] P.E., V. Lotoreichik, M. Tater: Smilansky model: a numerical analysis, *in preparation*

[BEKT15] D. Barseghyan, P.E., A. Khrabustovskyi, M. Tater: Spectral properties of Schrödinger operators with narrowing, below unbounded potential channels, *in preparation*

It remains to say

Thank you for your attention!