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On the Quasi-probability associated with the Weak Value

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Plan of Talk

1. Weak trajectory and quasiprobability

2. Physical value in HVT and quasiprobability

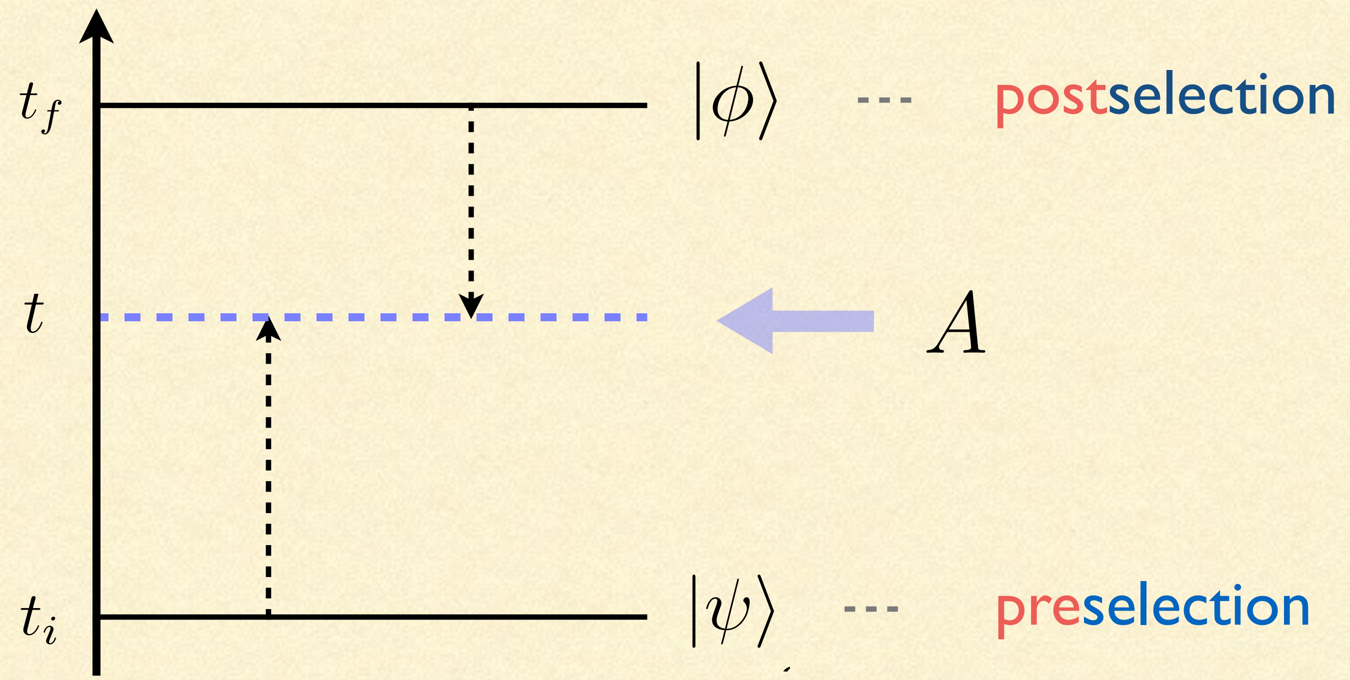
3. Postselected measurement and quasiprobability

1. Weak trajectory and quasiprobability

weak value as a new ‘observable’ of quantum mechanics

$$\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$$

weak value

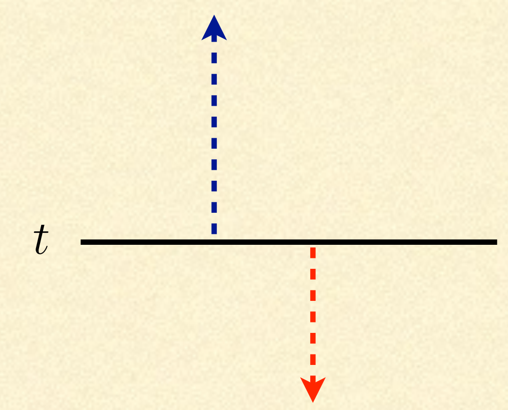


- useful for a deeper understanding of quantum phenomena
- applications for precision measurement

time-symmetric formulation of quantum mechanics

Y. Aharonov, P. G. Bergmann, J. L. Lebowitz (1964)

... the result of the measurement at t has implications not only for what happens after t but also for what happened in the past ...

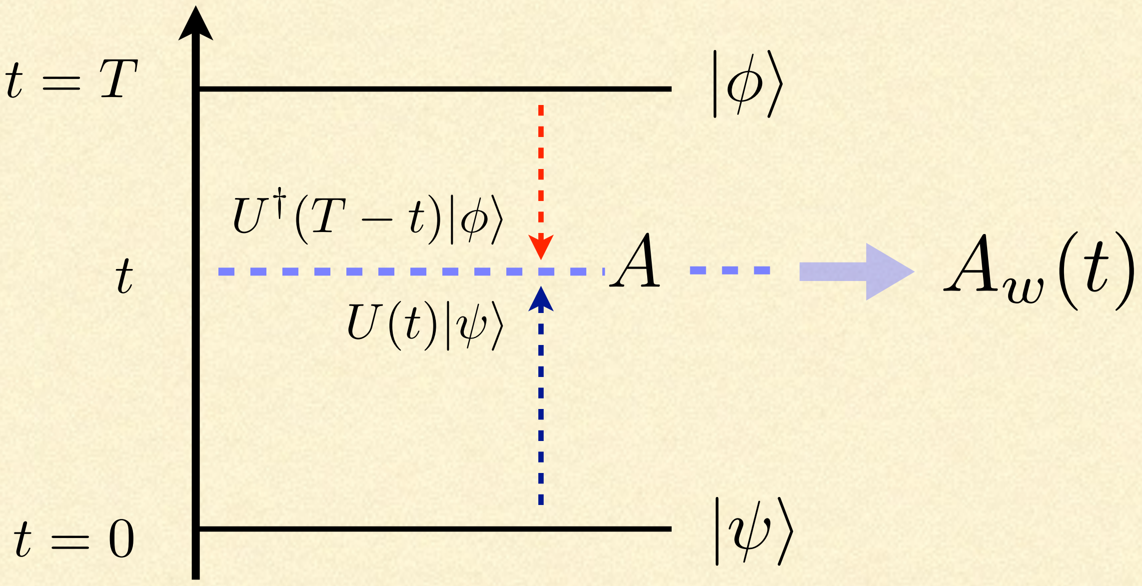


- consistent with all the predictions made by the standard description of QM
- shed new lights on quantum phenomena that were missed before (such as weak value, state reduction, tunneling etc.)
- may suggest generalizations of QM

Can the weak value offer an 'intuitive picture' ?

weak value under dynamical evolution

$$U(t) = \exp\left[-\frac{\mathrm{i}Ht}{\hbar}\right]$$



preselection: $U(t)|\psi\rangle$

postselection: $U^\dagger(T-t)|\phi\rangle$

time-dependent (dynamical) weak value

$$A_w(t) = \frac{\langle\phi|U(T-t)\cdot A\cdot U(t)|\psi\rangle}{\langle\phi|U(T-t)\cdot U(t)|\psi\rangle} = \frac{\langle\phi|U(T-t)AU(t)|\psi\rangle}{\langle\phi|U(T)|\psi\rangle}$$

like the expectation value, the dynamical weak value satisfies

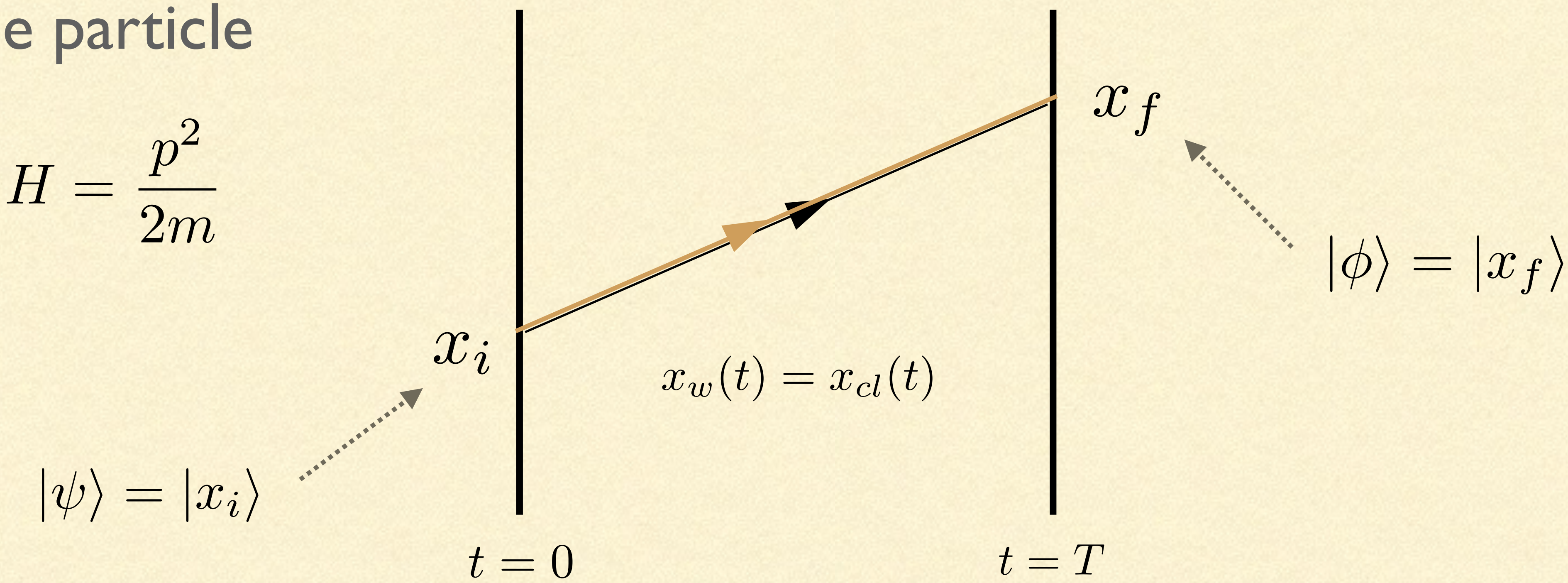
$$\begin{aligned}\frac{d}{dt}A_w(t) &= -\frac{\mathrm{i}}{\hbar}\frac{\langle\psi|U(T-t)\,[A,H]\,U(t)|\phi\rangle}{\langle\psi|U(T)|\phi\rangle} \\ &= -\frac{\mathrm{i}}{\hbar}[A,H]_w(t),\end{aligned}$$

Ehrenfest theorem for the weak value

trajectory in weak value -'weak trajectory' -

free particle

$$H = \frac{p^2}{2m}$$



$$x_w(t) = \frac{\langle\psi_f|U(T-t)xU(t)|\psi_i\rangle}{\langle\psi_f|U(T)|\psi_i\rangle}$$

trajectory of particle in terms of the weak value

$$= x_i + \frac{(x_f - x_i)t}{T} = x_{cl}(t)$$

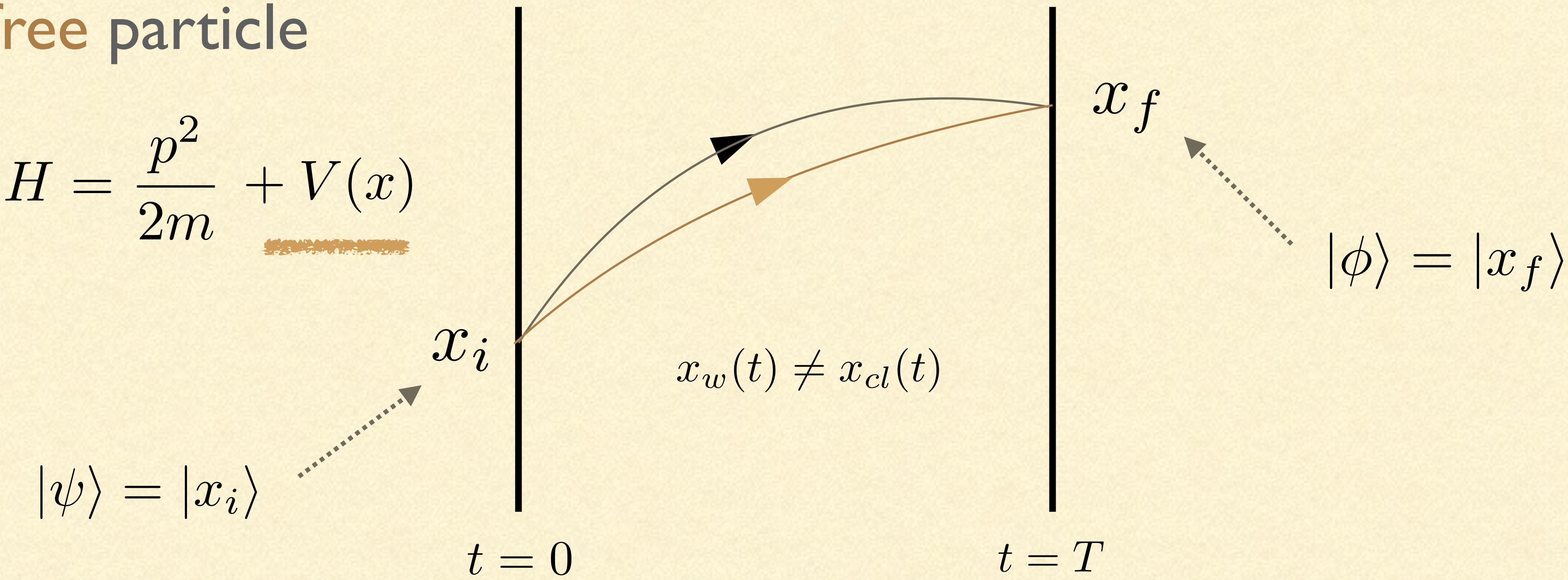
coincides with classical trajectory

up to quadratic potentials by Ehrenfest theorem

trajectory in weak value -'weak trajectory' -

non-free particle

$$H = \frac{p^2}{2m} + V(x)$$



$$x_w(t) = \frac{\langle \psi_f | U(T-t) x U(t) | \psi_i \rangle}{\langle \psi_f | U(T) | \psi_i \rangle}$$

trajectory of particle in terms of the weak value

semiclassical study of the trajectories

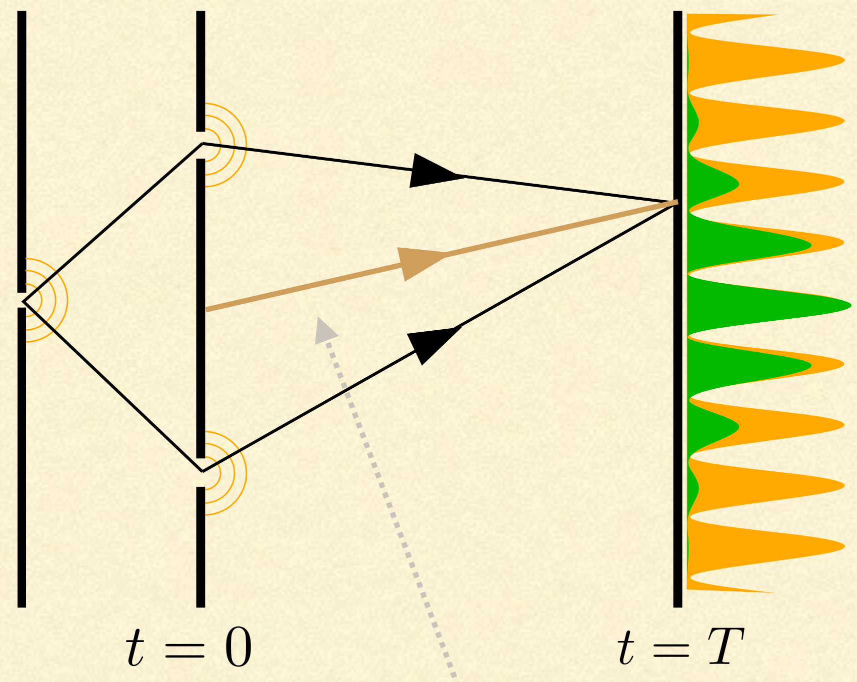
A. Tanaka, *PLA (2002)*

A. Matzkin, *PRL (2012)*

double slit experiment

T. Mori and I.T., *Quant. Stud. (2015)*

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|x_i\rangle + |-x_i\rangle\right)$$



$$|\phi\rangle = |x_f\rangle$$

$$U(t) = \exp\left[-\frac{\mathrm{i}Ht}{\hbar}\right]$$

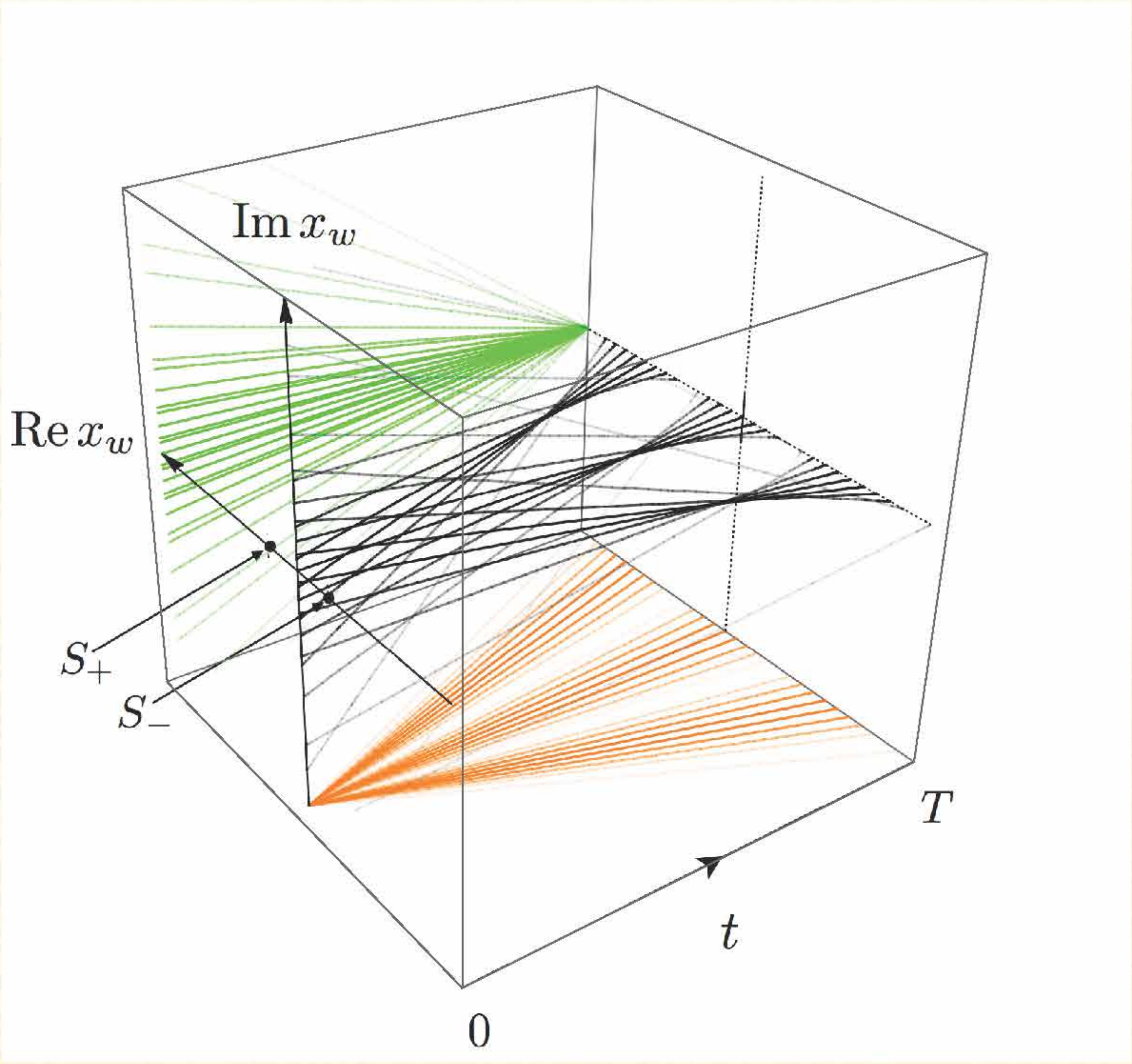
$$H = \frac{p^2}{2m}$$

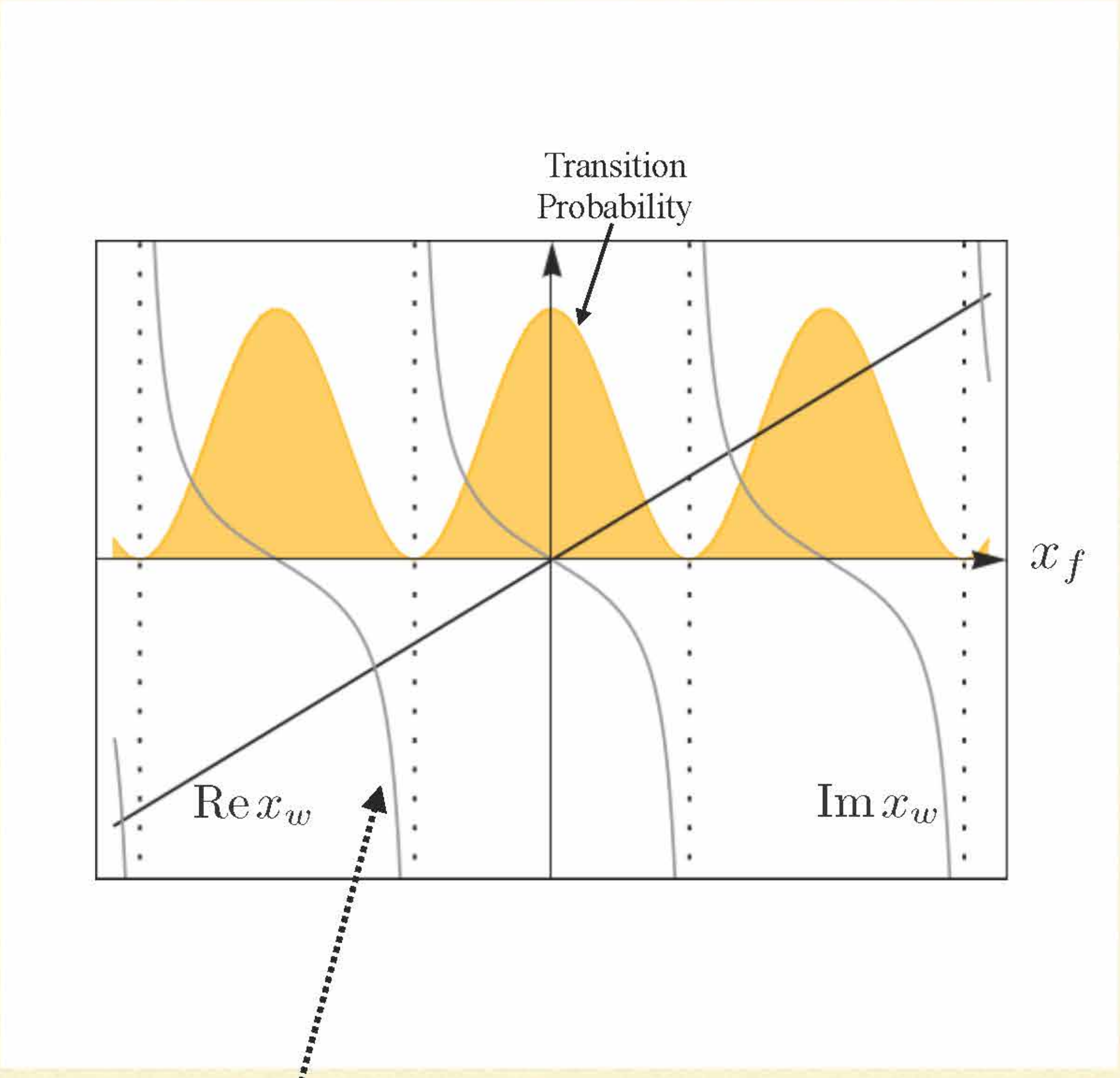
weak trajectory

$$x_w(t) = \frac{\langle\phi|U(T-t)xU(t)|\psi\rangle}{\langle\phi|U(T)|\psi\rangle} = \frac{x_f t}{T} + \mathrm{i}\frac{x_i(t-T)\tan\left(\frac{m}{\hbar}\frac{x_f x_i}{T}\right)}{T}$$

real imaginary

weak trajectories in complex space



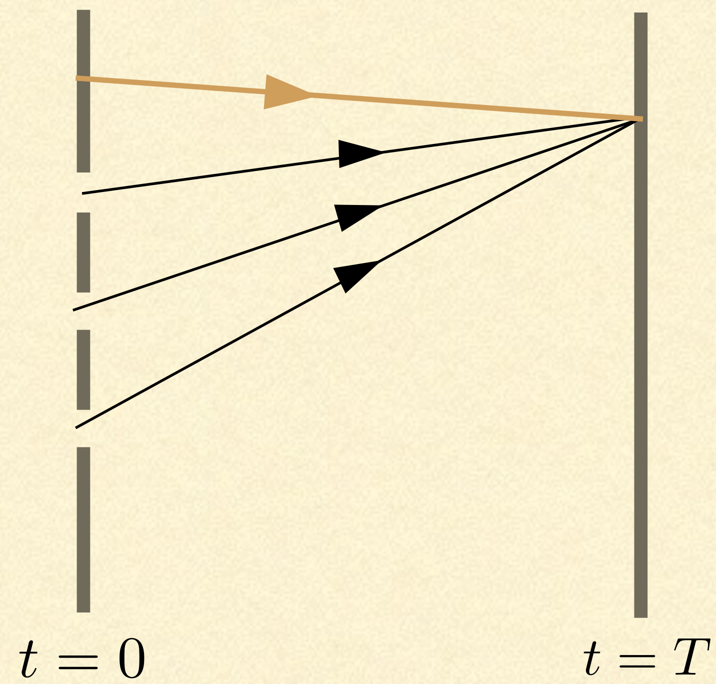


$\mathrm{Re}\, x_w(t)$ gives the 'average path' from the two slits

$\mathrm{Im}\, x_w(t)$ at $t = 0$ signifies interference effect

triple slit experiment

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|x_i\rangle + |0\rangle + |-x_i\rangle\right)$$



$$|\phi\rangle = |x_f\rangle$$

weak trajectory

$$x_w(t) = \frac{\langle\phi|U(T-t)xU(t)|\psi\rangle}{\langle\phi|U(T)|\psi\rangle} = x_f\frac{t}{T} + g(x_i,x_f)\left(1-\frac{t}{T}\right)$$

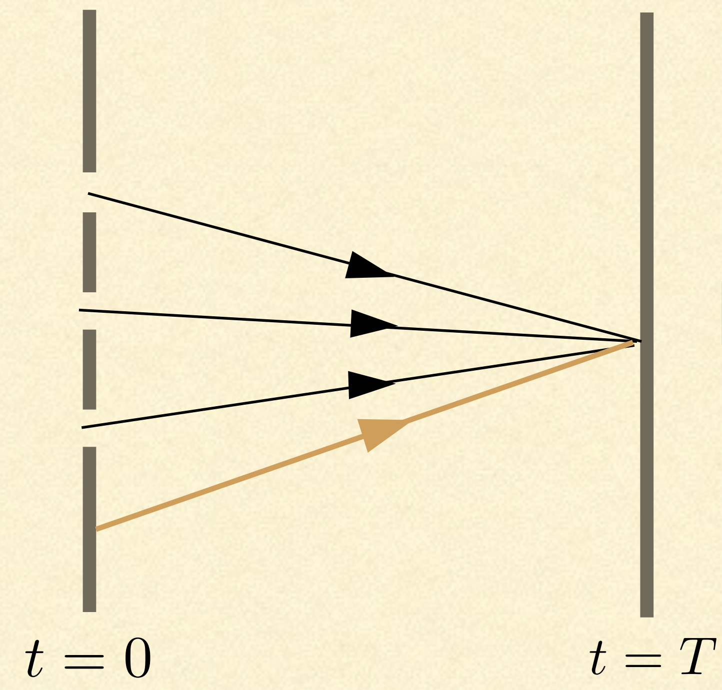
$$\mathrm{Re}\,g = \frac{2x_i\sin\left(\frac{m}{\hbar}\frac{x_fx_i}{T}\right)\sin\left(\frac{m}{\hbar}\frac{x_i^2}{2T}\right)}{3+2\cos\left(\frac{m}{\hbar}\frac{2x_fx_i}{T}\right)+4\cos\left(\frac{m}{\hbar}\frac{x_fx_i}{T}\right)\cos\left(\frac{m}{\hbar}\frac{x_i^2}{2T}\right)},$$

$$\mathrm{Im}\,g = -\frac{2x_i\left\{\cos\left(\frac{m}{\hbar}\frac{x_fx_i}{T}\right)+\cos\left(\frac{m}{\hbar}\frac{x_i^2}{2T}\right)\right\}\sin\left(\frac{m}{\hbar}\frac{x_fx_i}{T}\right)}{3+2\cos\left(\frac{m}{\hbar}\frac{2x_fx_i}{T}\right)+4\cos\left(\frac{m}{\hbar}\frac{x_fx_i}{T}\right)\cos\left(\frac{m}{\hbar}\frac{x_i^2}{2T}\right)},$$

still straight line
as function of time

triple slit experiment

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|x_i\rangle + |0\rangle + |-x_i\rangle\right)$$



$$|\phi\rangle = |x_f\rangle$$

weak trajectory

$$x_w(t) = \frac{\langle\phi|U(T-t)xU(t)|\psi\rangle}{\langle\phi|U(T)|\psi\rangle} = x_f\frac{t}{T} + g(x_i, x_f)\left(1 - \frac{t}{T}\right)$$

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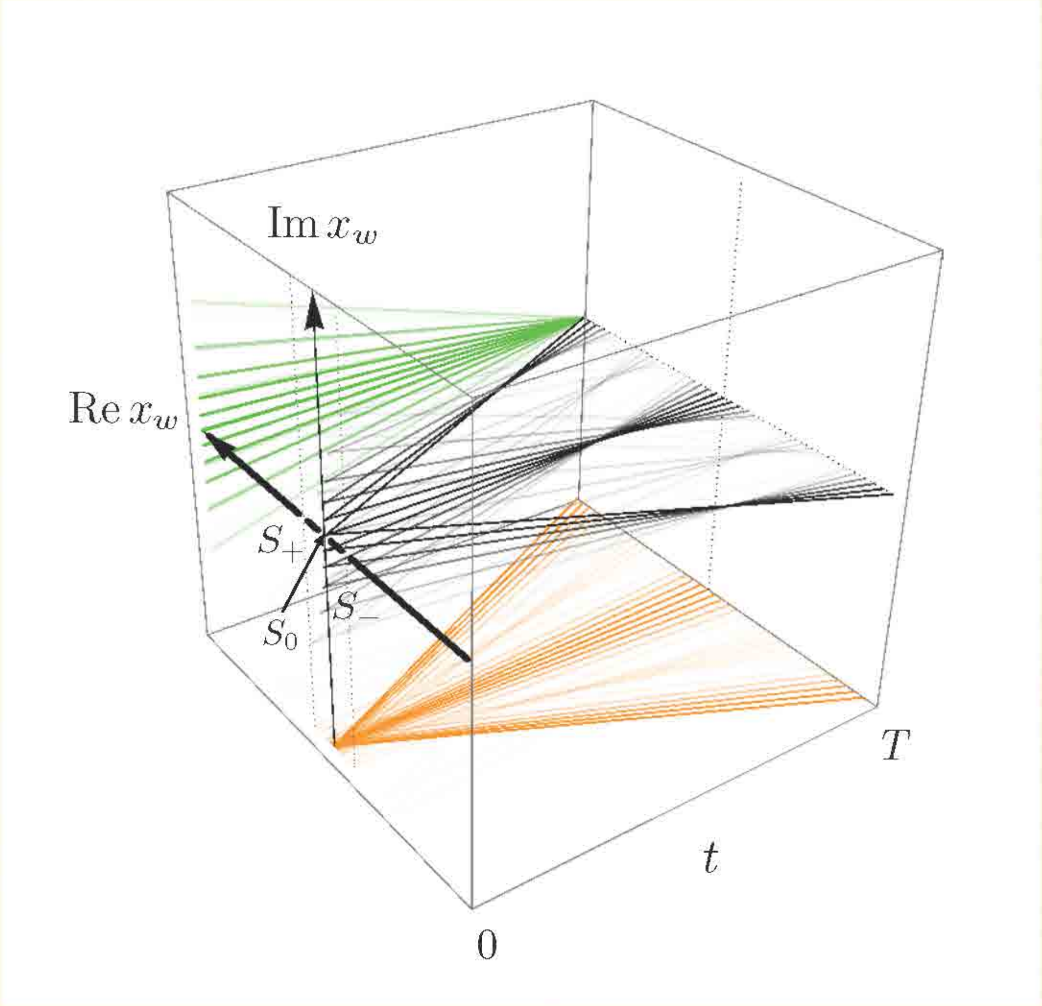
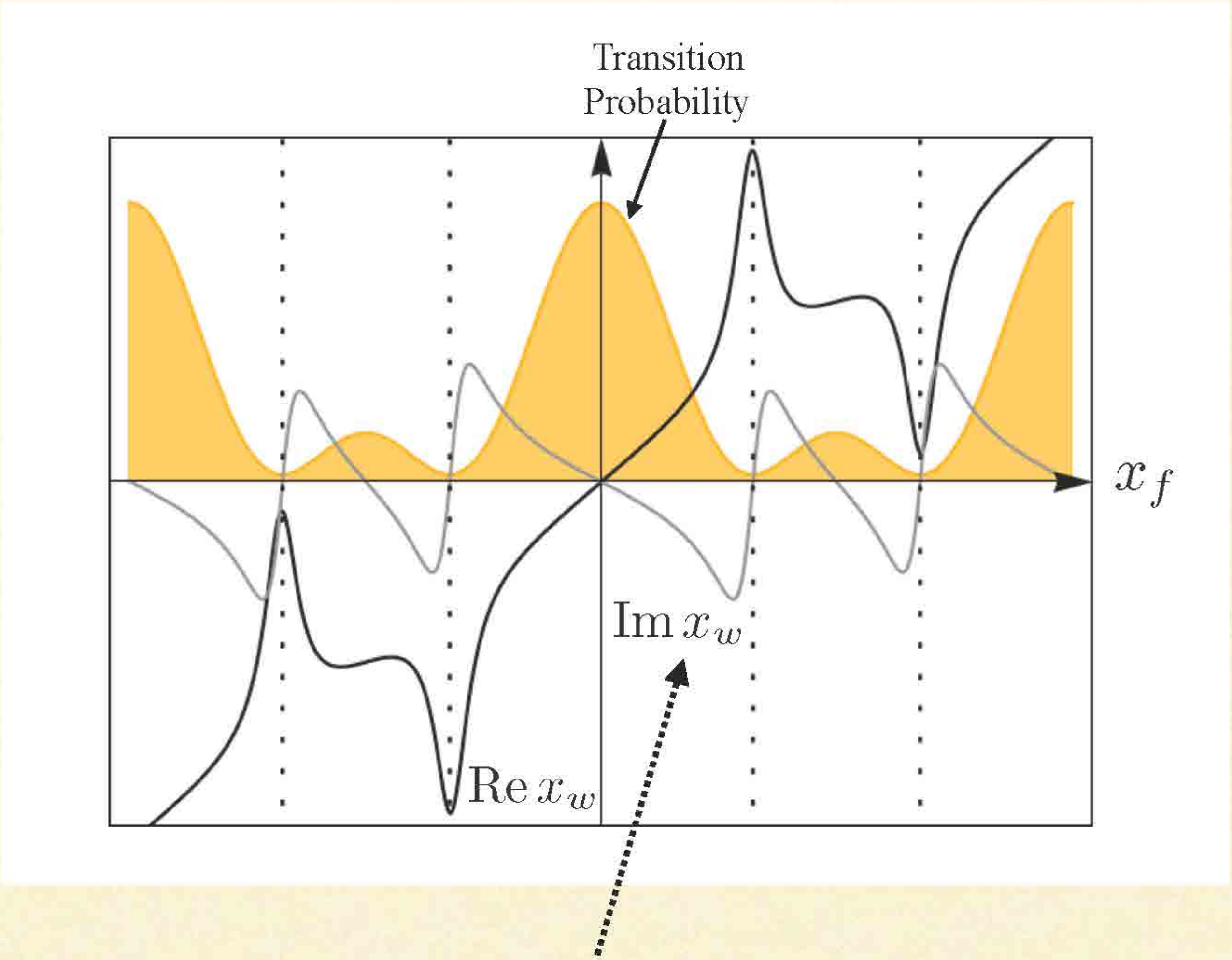
still straight line
as function of time

weak trajectories in complex space





$\mathrm{Im}\,x_w(t)$ **at** $t = 0$

signifies interference effect

multiple slit or general case

preselection

$$|\psi\rangle = \sum_{n=1}^{N} c_n |x_n\rangle, \qquad c_n \in \mathbb{C}$$

postselection

$$|\phi\rangle = |x_f\rangle.$$

weak trajectory

$$x_w(t) = \frac{\langle\phi|U(T-t)xU(t)|\psi\rangle}{\langle\phi|U(T)|\psi\rangle} = \sum_{n=1}^{N} \omega_n \, x_w^{x_n \to x_f}(t)$$

with 'weak quasiprobability'

$$\omega_n = \frac{c_n \langle x_f|U(T)|x_n\rangle}{\sum_n c_n \langle x_f|U(T)|x_n\rangle} \longrightarrow \frac{\langle\phi(T)|x_n\rangle\langle x_n|\psi\rangle}{\langle\phi(T)|\psi\rangle} = \frac{\langle\phi(T)|E_n^x|\psi\rangle}{\langle\phi(T)|\psi\rangle}$$

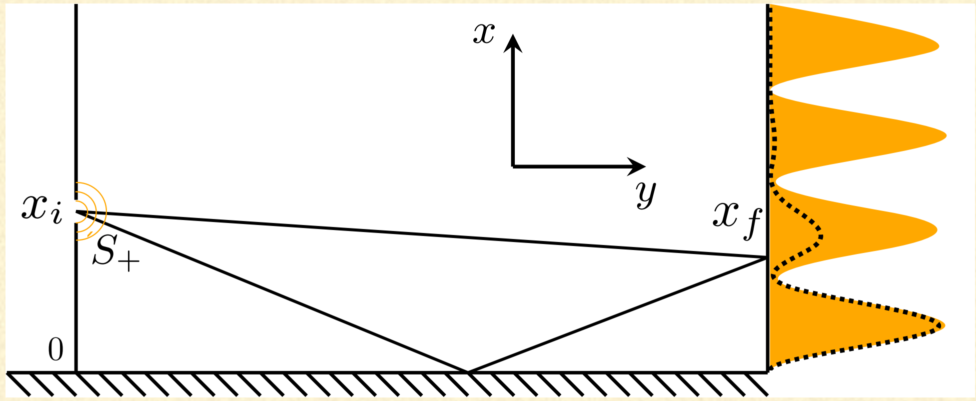
$$c_n \to \langle x_n|\psi\rangle$$

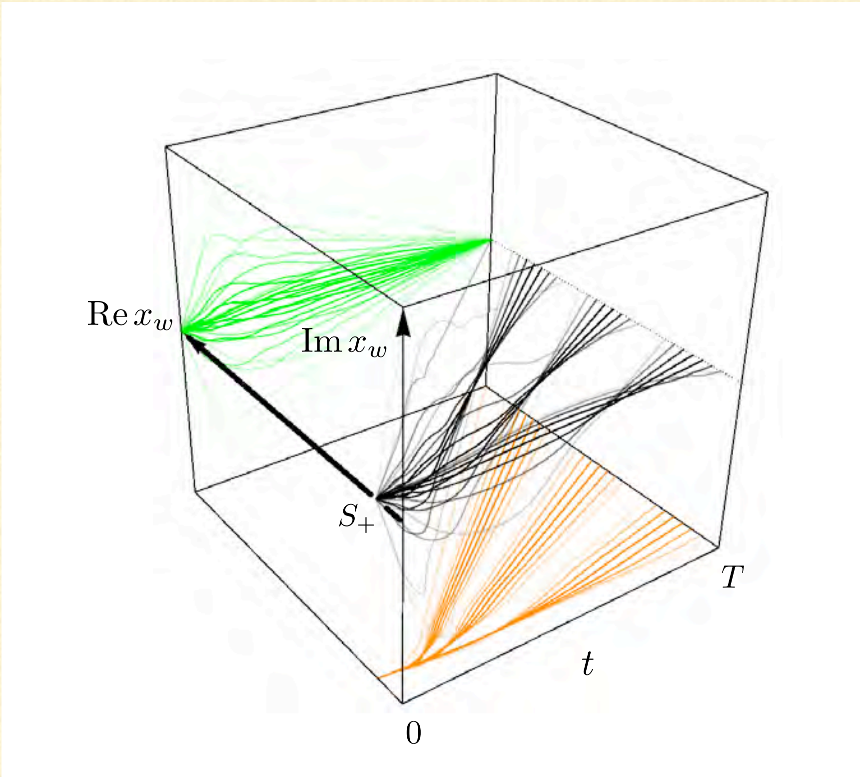
$$\langle\phi(T)| = \langle\phi|U(T)$$

what we have learned so far:

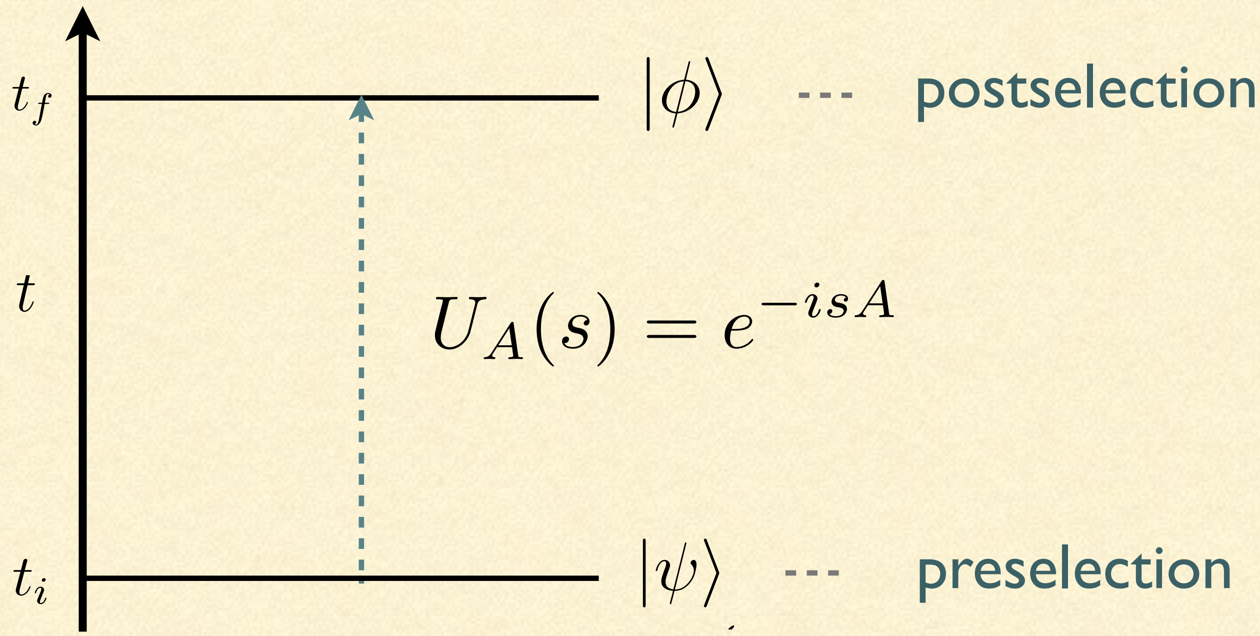
- it is given by the average over respective 'classical' weak trajectories weighted with 'weak quasiprobability'
- imaginary part describes the degree of interference (more next)

further example: Lloyd's mirror





weak value as a correction to transition probability



transition amplitude

$$K(s) := \langle \phi | U_A(s) | \psi \rangle$$

transition probability

$$P(s) = |K(s)|^2$$

relative change due to the unitary action (for small $s|A|$) Dressel et al., *RMP (2014)*

$$\frac{P(s)}{P(0)} = \frac{|\langle \phi | (1 - isA - (s^2/2)A^2 + \cdots) | \psi \rangle|^2}{|\langle \phi | \psi \rangle|^2}$$

$$= 1 + 2s\mathrm{Im}A_w + s^2 \left\{ |A_w|^2 - \mathrm{Re}(A^2)_w \right\} + \mathcal{O}(s^3)$$

imaginary part relates to interference

Dressel, Jordan, *PRA (2012)*
T. Mori, I. T., *PTEP (2015)*

transition amplitude through intermediate states

$$K_k(s) = \langle\phi|U_A(s)|\chi_k\rangle\langle\chi_k|\psi\rangle$$

intermediate states $\mathbb{I} = \sum_k |\chi_k\rangle\langle\chi_k|$

total probability

$$P(s) = \sum_k P_k(s) + \sum_{j\neq k} K_k(s)K_j^*(s)$$

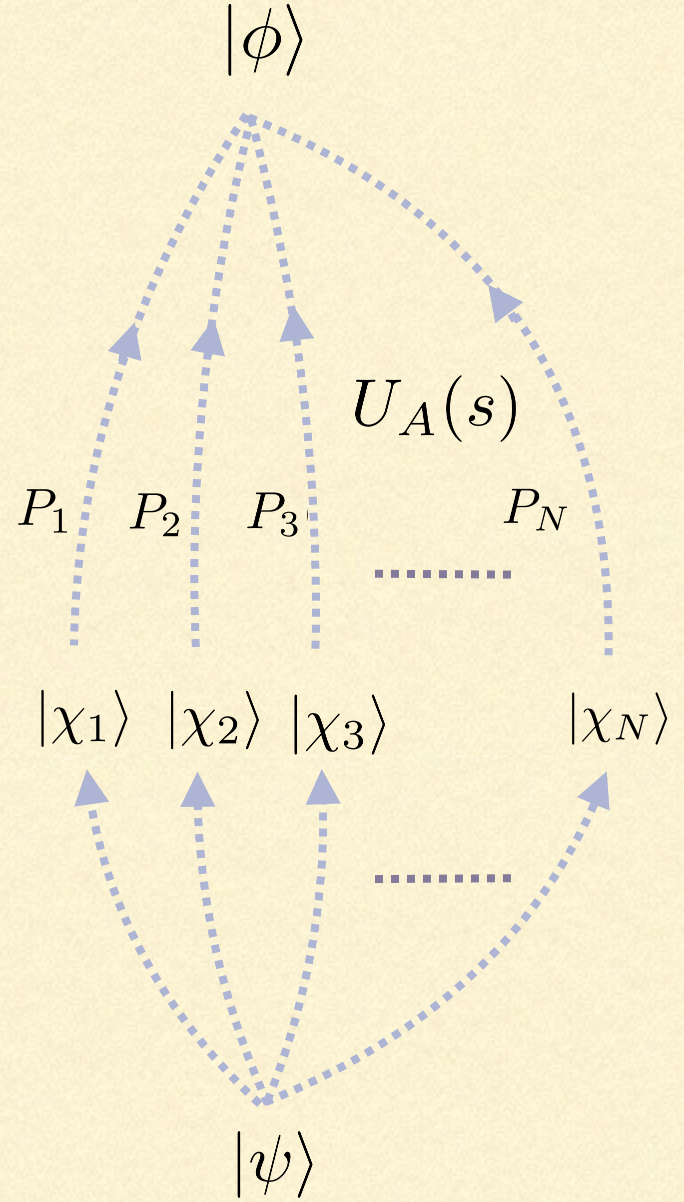
'off-diagonal'

'intensity' of interference

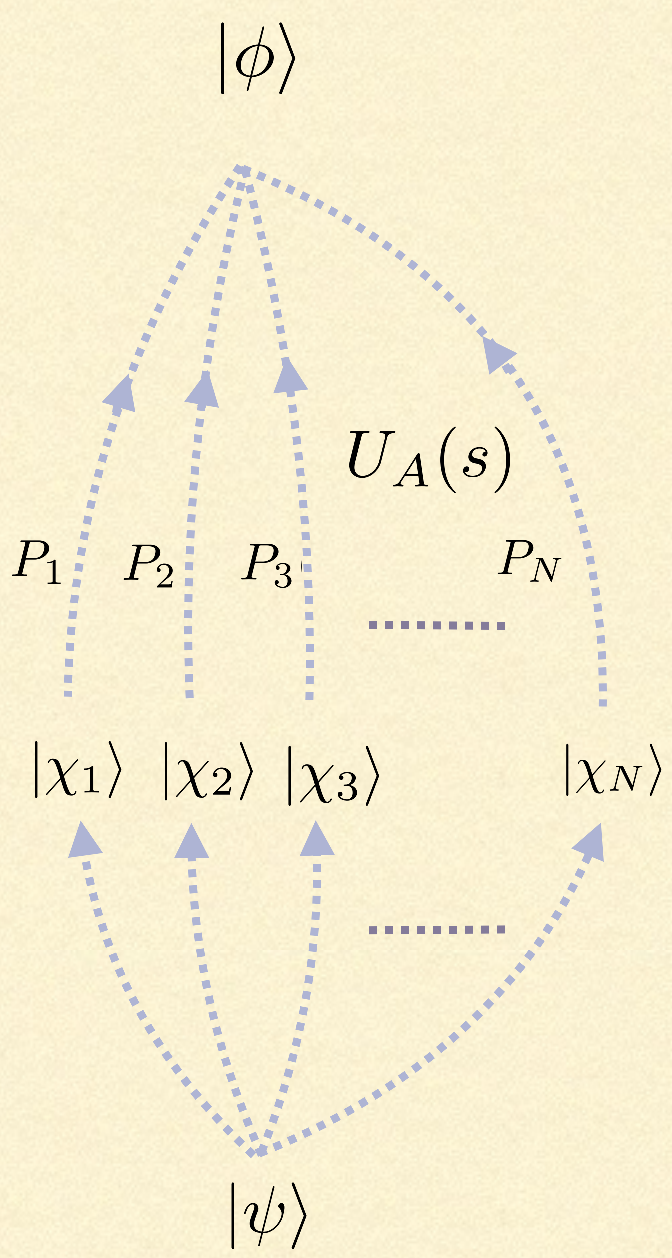
$$\mathcal{I} := \frac{1}{2}\lim_{s\to 0}\frac{1}{P(s)}\frac{\partial}{\partial s}\left[P(s) - \sum_k P_k(s)\right]$$

$$= \mathrm{Im}\left[A_w - \sum_k \frac{P_k(0)}{P(0)}A_w^k\right]$$

$$A_w^k = \frac{\langle\phi|A|\chi_k\rangle}{\langle\phi|\chi_k\rangle}$$



equivalent picture



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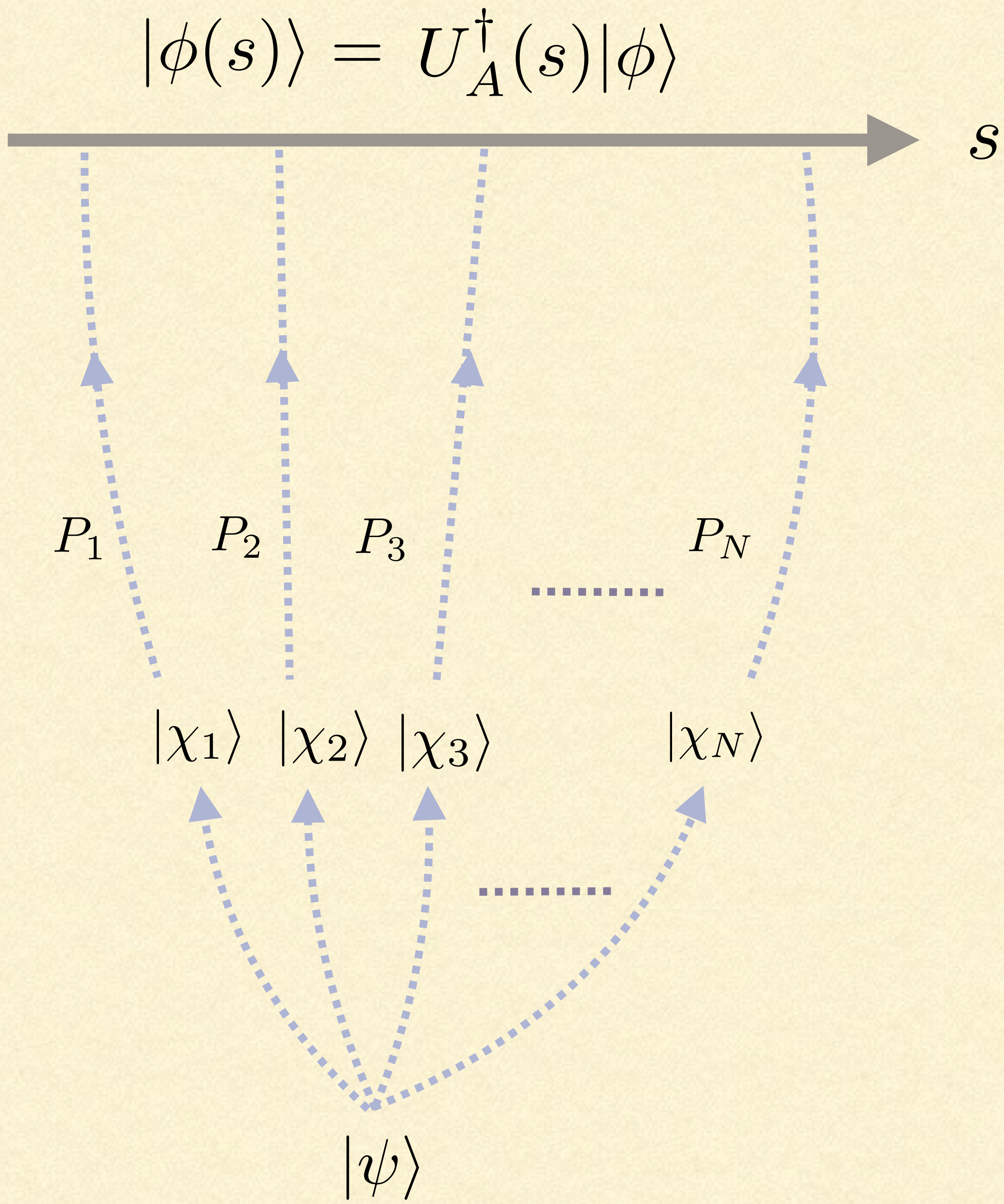
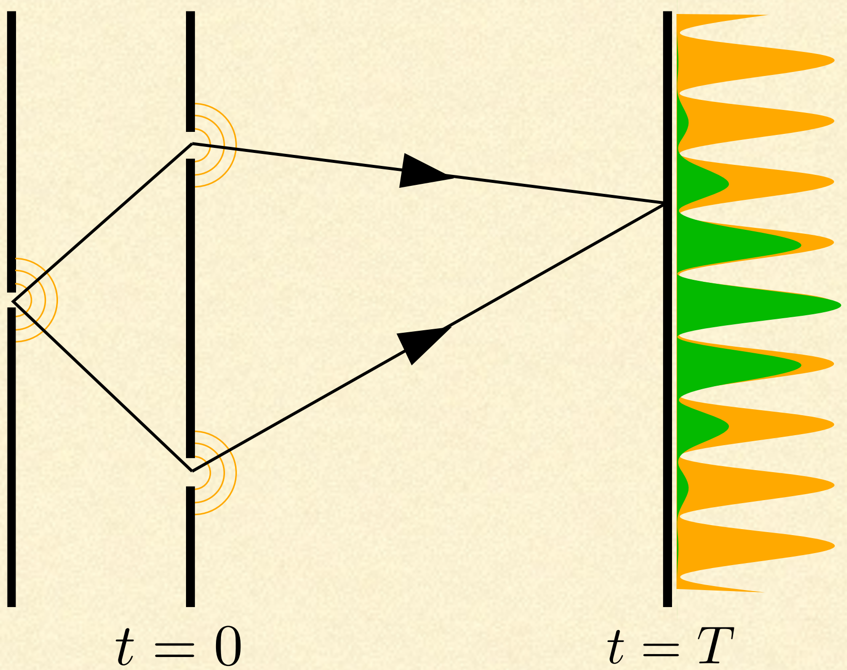
unitary family of postselections



ex.) double slit experiment



A → p generator of translation

$$|\phi(s)\rangle = e^{isp}|\phi\rangle$$

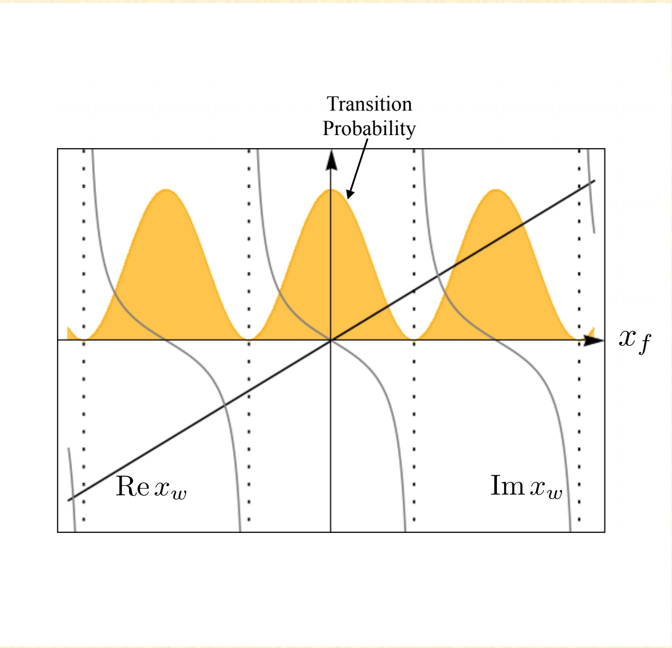
$$= e^{isp}|x_f\rangle = |x_f - s\rangle$$

unitary family of postselections

'intensity' of interference

$$\mathcal{I} = \operatorname{Im} p_w = m\frac{x_i \tan\left(\frac{m}{\hbar}\frac{x_f x_i}{T}\right)}{T} \quad \propto \operatorname{Im} x_w$$

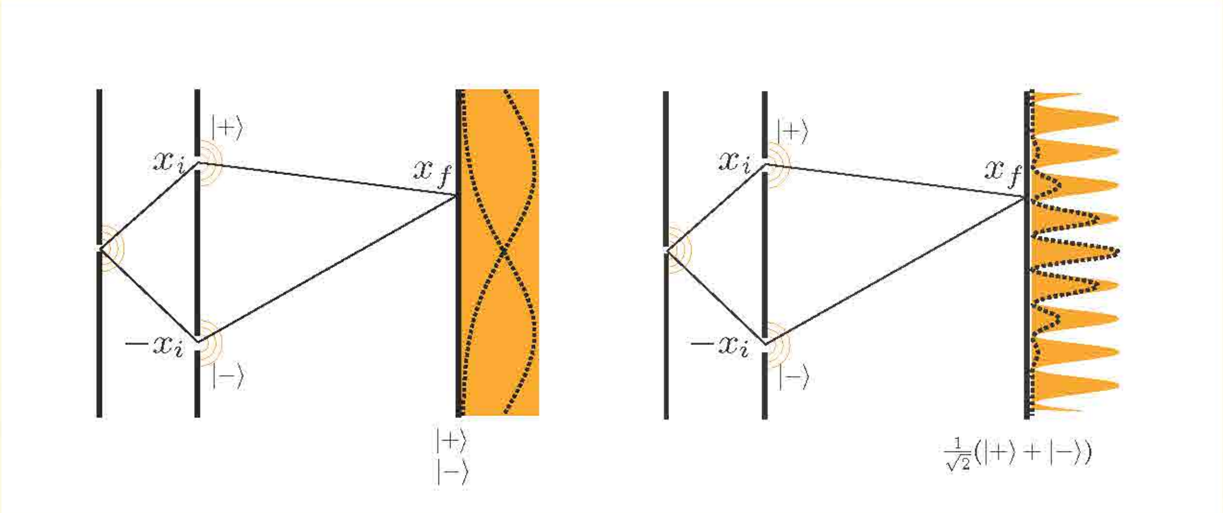
↑ Ehrenfest theorem



‘which path’ experiment

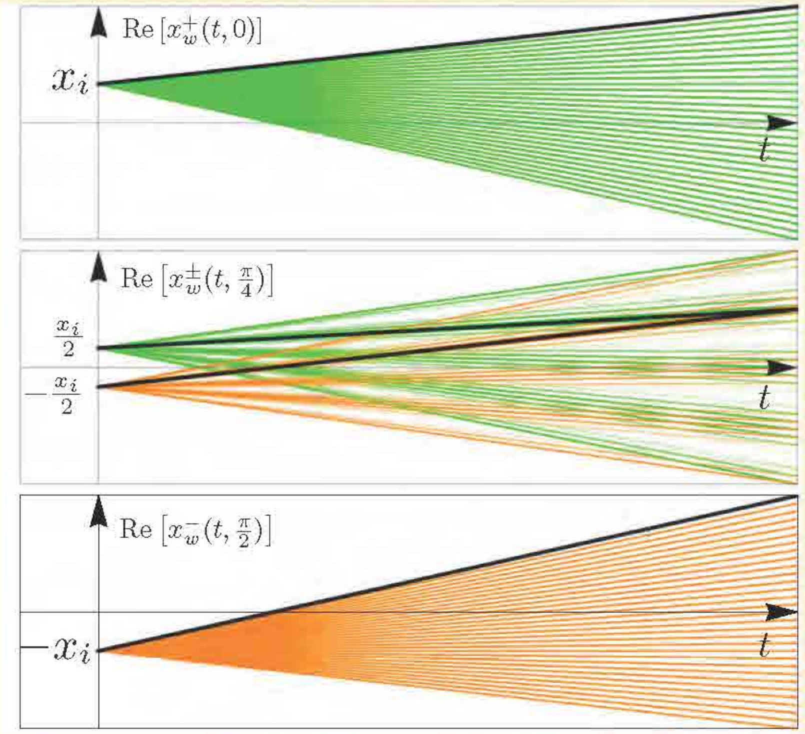
add spin degrees of freedom to obtain which path information

$$x^{\pm} = x \otimes |\pm\rangle\langle\pm|$$



preselection

$$|\psi_i\rangle = \frac{|x_i\rangle \otimes |+\rangle + |-x_i\rangle \otimes |-\rangle}{\sqrt{2}}$$



postselection

$$|\psi_f(\theta)\rangle = \frac{|x_f\rangle \otimes (\cos\theta|+\rangle + \mathrm{i}\sin\theta|-\rangle)}{\sqrt{2}}$$

weak trajectory from x_i

$$x_w^+(t,\theta) = \frac{\langle\psi_f(\theta)|U(T-t)x^+U(t)|\psi_i\rangle}{\langle\psi_f|U(T)|\psi_i\rangle} = \frac{\left[-x_i + (x_f + x_i)\frac{t}{T}\right]\sin\theta}{\sin\theta + e^{-\mathrm{i}\chi}\cos\theta}$$

$$\chi := \frac{2m}{\hbar}\frac{x_f x_i}{T}$$

2. Physical value in HVT and quasiprobability

expectation value vs weak value

expectation value

$$\langle A \rangle := \langle \psi | A | \psi \rangle$$

$$\langle A \rangle : \mathcal{H} \to R(A) \subset \mathbb{R}$$ 'one-state' value

within the (real) range of spectrum

weak value

$$A_w := \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle}$$

$$A_w : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$$ 'two-state' value

entire range of complex numbers

... the result of measurement of a spin component of spin 1/2 particle can turn out to be 100 ...

Y. Aharonov, D. Z. Albert, L. Vaidman (1988)

property of weak value

(analogous to expectation value)

$$A_w := \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$$

1) agrees with the eigenvalue if the preselected (or postselected) state is an eigenstate

$$A|\psi\rangle = a|\psi\rangle \quad \text{or} \quad A|\phi\rangle = a|\phi\rangle \longrightarrow A_w = a$$

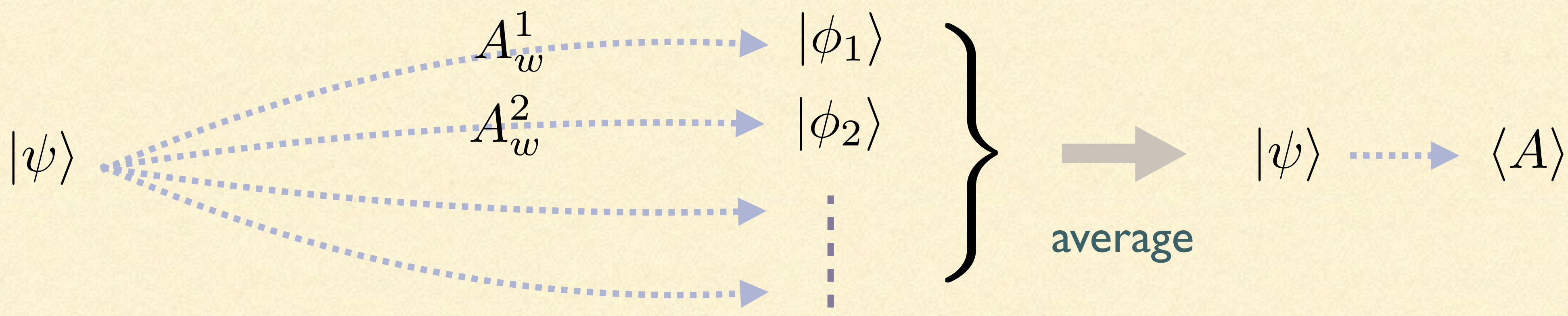
2) fulfills sum rule $A = B + C \longrightarrow A_w = B_w + C_w$

but does not fulfill product rule $A = B\,C \not\longrightarrow A_w = B_w\,C_w$

link between the weak value and the expectation value

$$\sum_k |\langle\phi_k|\psi\rangle|^2 \frac{\langle\phi_k|A|\psi\rangle}{\langle\phi_k|\psi\rangle} = \langle\psi|A|\psi\rangle$$

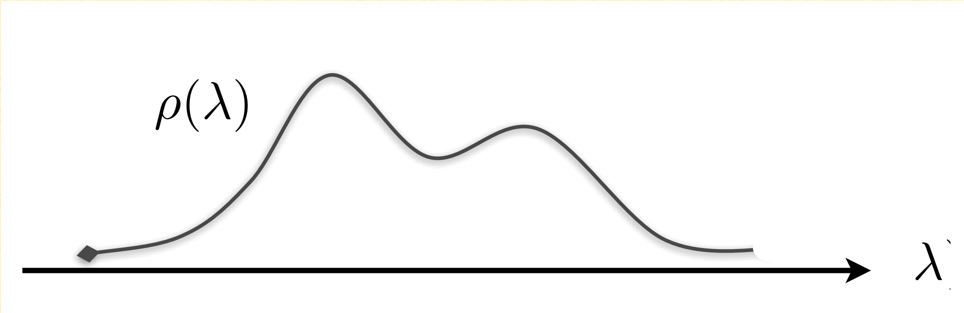
average of weak values over postselections



average of the weak values assigned to all possible processes gives the expectation value

“interpretation” as physical value in HVT (ontological model)

de Broglie-Bohm theory



- hidden variable $\lambda \longleftrightarrow x$
- probability distribution $\rho(\lambda) = |\psi(x)|^2$
- expectation value $\langle A \rangle_{dBB} = \int d\lambda\, \rho(\lambda) A(\lambda)$

if we require

$$\langle A \rangle_{dBB} = \langle A \rangle_{QM} = \langle \psi | A | \psi \rangle \quad \Longrightarrow \quad A(\lambda) = \mathrm{Re} \frac{\langle x | A | \psi \rangle}{\langle x | \psi \rangle}$$

‘local expectation value’

recall

$$\sum_k |\langle \phi_k|\psi\rangle|^2 \frac{\langle \phi_k|A|\psi\rangle}{\langle \phi_k|\psi\rangle} = \langle \psi|A|\psi\rangle$$

$|\phi_k\rangle \to |x\rangle$

$$\int dx\, |\psi(x)|^2 \frac{\langle x|A|\psi\rangle}{\langle x|\psi\rangle} = \langle \psi|A|\psi\rangle$$

or

$$\int d\lambda\, \rho(x)\, \mathrm{Re} \frac{\langle x|A|\psi\rangle}{\langle x|\psi\rangle} = \langle \psi|A|\psi\rangle$$

‘local expectation value’ in dBB theory

in general

probability of obtaining a_i in measuring observable A

$$p\,(A = a_i\,|\psi\,) = \langle \psi, E_i^A \psi \rangle \qquad E_i^A = |a_i\rangle\,\langle a_i|$$

if $E_j^B = |b_j\rangle\,\langle b_j|$ for some observable B

$$\begin{aligned}
p\,(A = a_i\,|\psi\,) &= \langle \psi, E_i^A \psi \rangle \\
&= \sum_{j=1}^{M} \langle \psi, E_j^B E_i^A \psi \rangle \\
&= \sum_{j=1}^{M} \frac{\langle b_j, E_i^A \psi \rangle}{\langle b_j, \psi \rangle}\,|\langle b_j, \psi \rangle|^2 \\
&= \sum_{j=1}^{M} c\,(A = a_i\,|\psi, b_j\,)\,p\,(B = b_j\,|\psi\,)
\end{aligned}$$

compare with

ontological model by Harrigan & Spekkens (2010)

$$p\,(A = a_i\,|\psi\,) = \sum_k p\,(A = a_i\,|\lambda_k\,)\,p\,(\lambda_k\,|\psi\,)$$ 'ontic states' λ_k

$$\longrightarrow \quad c\,(A = a_i\,|\psi, \phi\,) := \frac{\langle \phi, E_i^A \psi \rangle}{\langle \phi, \psi \rangle}$$ 'weak quasiprobability'

appears naturally in the ontological interpretation of QM, modulo the state dependence (and complex-valuedness)

properties

M. Ozawa, AIP Conf. Proc. (2011)

A. Steinberg, Phys. Rev. A (1995)

- complex weak quasiprobability

$$a_i \mapsto c\,(A = a_i\,|\psi, \phi\,) \qquad \sum_i c\,(A = a_i\,|\psi, \phi\,) = 1$$

- reduction to probability

$$c\,(A = a_i\,|\psi, \psi\,) = \frac{\langle \psi, E_i^A \psi \rangle}{\langle \psi, \psi \rangle} = \langle \psi, E_i^A \psi \rangle = p\,(A = a_i\,|\psi\,)$$

- relation to joint quasiprobability (Kirkwood function)

$$c\,(A = a_i\,|\psi, b_i\,) = \frac{q\,(A = a_i,\ B = b_j\,|\psi\,)}{p\,(B = b_i\,|\psi\,)} \longleftarrow \langle \psi, E_j^B E_i^A \psi \rangle$$

- weak value as expectation value

$$\sum_i a_i c\,(A = a_i\,|\phi, \psi\,) = A_w$$

3. Postselected measurement and quasiprobability

complex probability measure (extending Gleason)

$\mu : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$

T. Morita and I.T. (2012)

- $\mu(\mathbb{1}) = 1$ $\qquad dim(\mathcal{H}) \geq 3$
- $\mu\left(\sum_i E_i\right) = \sum_i \mu(E_i)$ $\qquad \{E_i\}$ mutually orthogonal

$$\mu(E) = \mathrm{tr}(WE)$$

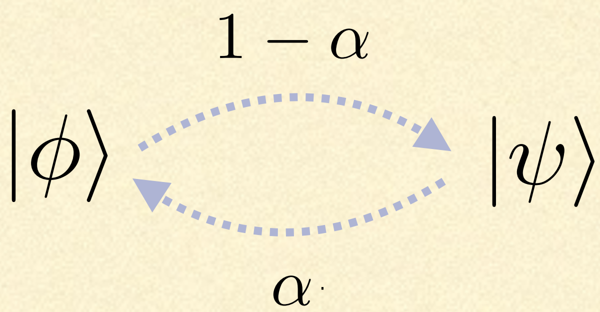
W trace-class

$$= \alpha \frac{\langle\phi|E|\psi\rangle}{\langle\phi|\psi\rangle} + (1-\alpha)\frac{\langle\psi|E|\phi\rangle}{\langle\psi|\phi\rangle}$$

preselection
postselection

arbitrariness $\alpha \in \mathbb{C}$

physical observable



$$A = \sum_i a_i E_i^A$$

$$\lambda(A) = \sum_i a_i \mu(E_i^A) = \mathrm{tr}(WA)$$

$$= \alpha \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle} + (1-\alpha)\frac{\langle\psi|A|\phi\rangle}{\langle\psi|\phi\rangle}$$

Note: most of the properties of the weak value (and the quasi probability) mentioned earlier hold even with the parameter α !

family of joint quasiprobabilities

J. Lee and I.T., in preparation

distribution for two (non-commuting) observables A, B

$$u^{\alpha}_{AB}[\phi](s,t) := \langle \phi, e^{i(1-\alpha)sA} e^{itB} e^{i\alpha sA} \phi \rangle \qquad \alpha \in \mathbb{R}$$

$\mathscr{F}$ ↓ Fourier transformation

$$w^{\alpha}_{AB}[\phi](a,b) := \int_{\mathbb{R}^2} e^{-i(as+bt)} u^{\alpha}_{AB}[\phi](s,t)\ dm_2(s,t)$$

then

$$dm_n(x) := (2\pi)^{-n/2} d\beta_n(x)$$

renormalized n-dimensional Lebesgue measure

$$\int_{\mathbb{R}^2} w^{\alpha}_{AB}[\phi](a,b)\ dm_2(a,b) \ = u^{\alpha}_{AB}[\phi](-0,-0) \ = \|\phi\|^2 = 1$$

joint quasiprobability

marginals

$$\mathrm{P}[A = a; w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}} w^{\alpha}_{AB}[\phi](a,b)\ dm(b)\ = \mu^{\phi}_{A}(a)$$

$$\mathrm{P}[B = a; w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}} w^{\alpha}_{AB}[\phi](a,b)\ dm(a)\ = \mu^{\phi}_{B}(b)$$

with

$$\mu^{\phi}_{X}(B) := \langle \phi, E_X(B)\phi \rangle$$ probability measure

expectation value

$$\mathrm{E}[A; w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}} a\ w^{\alpha}_{AB}[\phi](a,b) dm_2(a,b)$$

$$= (1-\alpha)\langle \phi, A\phi \rangle + \alpha \langle \phi, A\phi \rangle\ =\ \langle \phi, A\phi \rangle$$

$$\mathrm{E}[B; w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}} b\ w^{\alpha}_{AB}[\phi](a,b) dm_2(a,b) = \langle \phi, B\phi \rangle$$

covariance

$$\mathrm{CV}[A, B; w^{\alpha}_{AB}[\phi]] := \mathrm{E}[(A - \mathrm{E}[A; w^{\alpha}_{AB}[\phi]])(B - \mathrm{E}[B; w^{\alpha}_{AB}[\phi]]); w^{\alpha}_{AB}[\phi]]$$
$$= \mathrm{E}[AB; w^{\alpha}_{AB}[\phi]] - \mathrm{E}[A; w^{\alpha}_{AB}[\phi]] \cdot \mathrm{E}[B; w^{\alpha}_{AB}[\phi]],$$

with

$$\mathrm{E}[AB; w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}^2} ab \; w^{\alpha}_{AB}[\phi](a, b) dm_2(a, b)$$
$$= (1 - \alpha)\langle \phi, AB\phi \rangle + \alpha \langle \phi, BA\phi \rangle$$

mixture of ordering

at $\alpha = 1/2$, it reduces to

$$\mathrm{CV}[A, B; w^{1/2}_{AB}[\phi]] = \mathbb{E}[(AB + BA)/2; \phi] - \mathbb{E}[A; \phi] \cdot \mathbb{E}[B; \phi]$$
$$= \mathbb{CV}[A, B; \phi],$$

quantum covariance

relation to previously known quasiprobability distributions

- Wigner function

$\mathcal{H} = L^2(\mathbb{R}^2),\ A = \hat{p},\ B = \hat{x}$ and $\alpha = 1/2$

$$w^{1/2}_{\hat{p}\hat{x}}[\psi](p,x) = \int_{\mathbb{R}} \overline{\psi(x+y/2)}\psi(x-y/2)e^{ixy}\, dm(y) \ = W^{\psi}(x,p)$$

→ $\alpha = 1/2$ gives the unique case when $w^{\alpha}_{AB}[\phi](a,b)$ becomes real

- Kirkwood function

$\alpha = 1.$ (or $\alpha = 0$)

$$w^{1}_{AB}[\phi](a,b) = \langle \phi, b\rangle \langle b, a\rangle \langle a, \phi\rangle \ = K(a,b;\phi)$$

conditional quasiprobability relevant to ‘postselected measurement’

$$\mathrm{CP}[A=a|B=b;w^{\alpha}_{AB}[\phi]] := \frac{w^{\alpha}_{AB}[\phi](a,b)}{\mathrm{P}[B=b;w^{\alpha}_{AB}[\phi]]} = \frac{w^{\alpha}_{AB}[\phi](a,b)}{|\langle b,\phi\rangle|^2}$$

$|\phi\rangle$ preselection

$|b\rangle$ postselection

quasi-expectation value

$$\mathrm{CE}[A|B=b;w^{\alpha}_{AB}[\phi]] := \int_{\mathbb{R}} a\, d\mathrm{CP}[A=a|B=b;w^{\alpha}_{AB}[\phi]]$$

$$= \alpha\frac{\langle b, A\phi\rangle}{\langle b,\phi\rangle} + (1-\alpha)\frac{\overline{\langle b, A\phi\rangle}}{\langle b,\phi\rangle}$$

in agreement with

$$\lambda(A) = \alpha\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle} + (1-\alpha)\frac{\langle\psi|A|\phi\rangle}{\langle\psi|\phi\rangle}$$

representation in terms of probability

$$w_{AB}^{\alpha}[\phi](a,b) = \left(\overline{\langle b, E_A(\cdot)\phi\rangle}_{(1-\alpha)} * \langle b, E_A(\cdot)\phi\rangle_{\alpha}\right)(a)$$

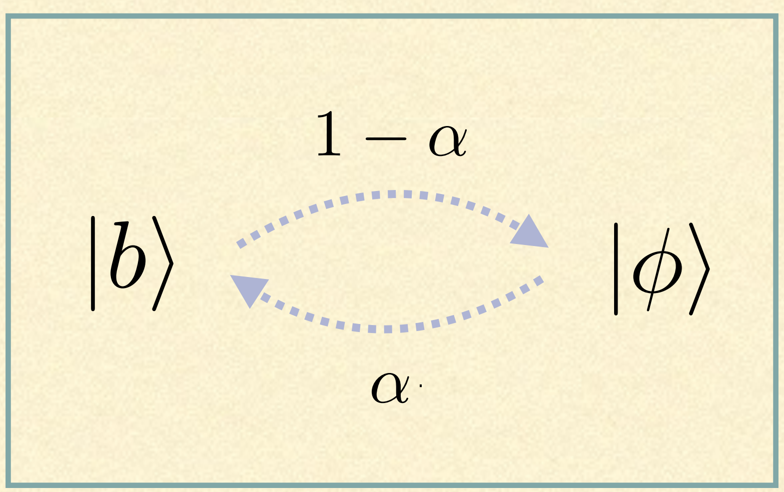
with

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\; dm_n(y)$$

convolution

$$f_t(x) := |t|^{-n} f\left(\frac{x}{t}\right)$$

scaling



Concluding Remarks

- The weak value may be regarded as the average of the 'classical weak values' with respect to the (conditional) weak quasiprobability associated with the given transition processes. The imaginary part describes the degree of interference involved in the processes.
- The weak quasiprobability (or the weak value) has a natural position in HVT (ontological model) when complexity is allowed. It admits an arbitrary parameter α, which is related to the ratio of mixture between the forward and backward processes.
- The joint quasiprobability, which may be relevant to 'postselected measurement' in the conditional form, admits a family containing the Wigner function ($\alpha = 1/2$) and the Kirkwood function ($\alpha = 0, 1$).

In all aspects, quasiprobability lies at the heart of the weak value and, possibly, at the heart of quantum mechanics.

Thank you!