A topological formulation for exotic quantum holonomy

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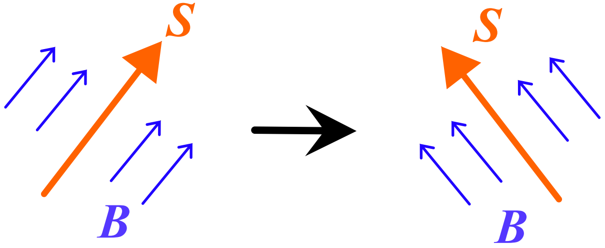
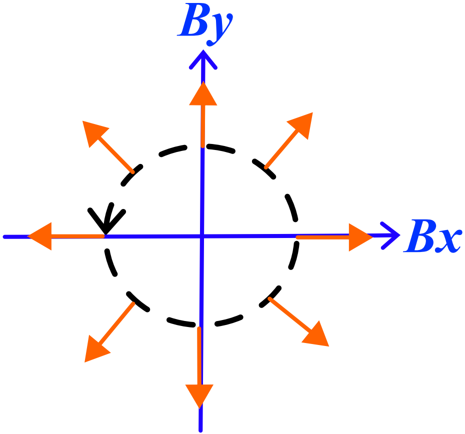
Second Dynamics Days Central Asia @ Khiva, Uzbekistan

Example: adiabatic response of spin $\boldsymbol{S}$ to the classical magnetic field $\boldsymbol{B}$



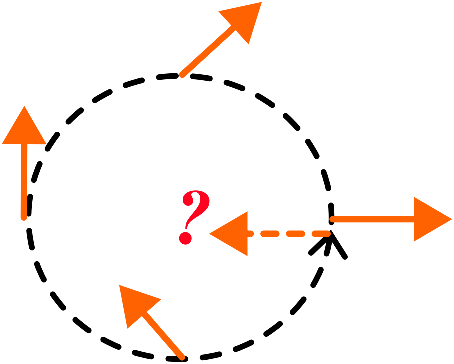
Suppose $\boldsymbol{S}$ is initially aligned to $\boldsymbol{B}$. As the direction of $\boldsymbol{B}$ is changed *gently*, $\boldsymbol{S}$ follows the direction of $\boldsymbol{B}$, according to the adiabatic theorem.

A quasi-static adiabatic cycle in $\boldsymbol{B}$-space



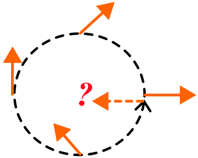
The spin comes back to the original direction after the completion of any adiabatic cycle in $\boldsymbol{B}$-space (i.e., the absence of exotic quantum holonomy).

Any exotic adiabatic cycle flips the spin?

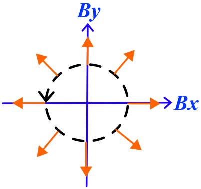


Exotic quantum holonomy

The change induced such an exotic cycles is called exotic quantum holonomy (a.k.a. Cheon's eigenspace anholonomy).



The term *holonomy* is derived from the phase holonomy (a.k.a. geometric phase, or, Berry phase, or, the molecular Aharonov-Bohm effect).



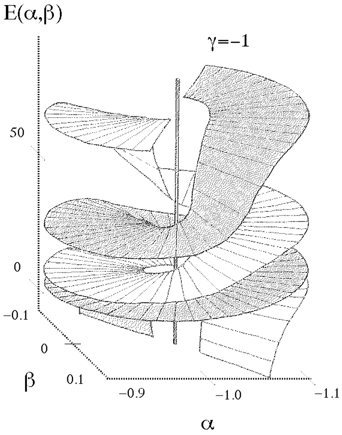
Aim

1. Provide an outline of the exotic quantum holonomy
2. Explain a topological formulation of the exotic quantum holonomy

Ref. AT and T. Cheon, `arXiv:1402.1634` (Phys. Lett. A, **379** (2015) p.1693) and references therein.

The first example

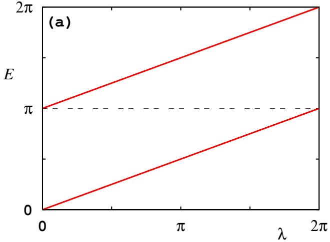
T. Cheon, PLA **248** (1998).



Eigenenergies of a particle in a 1-dim. box under *a generalized point potential*, which has two parameters α and β.

The minimal example — in a quantum kicked spin-$\frac{1}{2}$

AT and M. Miyamoto, PRL **98** (2007).



Möbius strip made of quasienergies

The spin is under periodic pulses whose strength is λ:

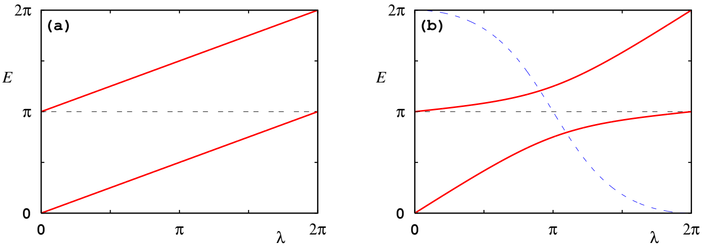
$$H(t) = \pi\frac{1+\sigma_z}{2} + \lambda\frac{1+\sigma_x}{2}\sum_{n=-\infty}^{\infty}\delta(t-n).$$

Floquet operator of the kicked spin model

The Floquet operator, which describes the time evolution during a period, of the kicked spin is

$$U(\lambda) = U_0 e^{-i\lambda|v\rangle\langle v|},$$

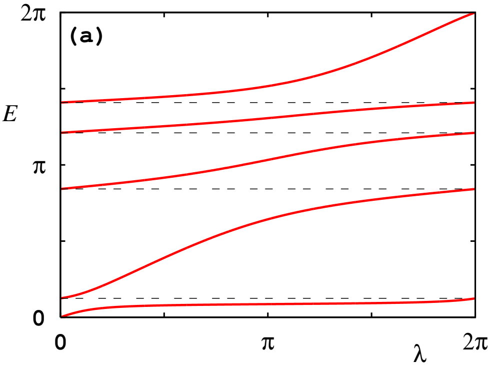
where $|v\rangle$ is a normalized vector, and U_0 describes the unitary time evolution of unperturbed system (Combescure, JSP **59** (1990); Milek and Seba, PRA **42**, 1990).



The eigenspace anholonomy occurs for a generic choice of U_0 and $|v\rangle$.

Multi-level quantum maps

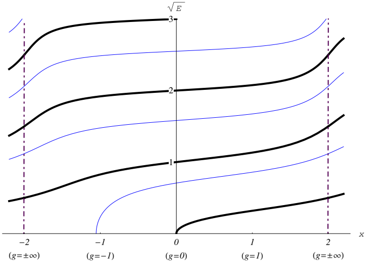
M. Miyamoto and AT, PRA **76** (2007).



Quasienergy anholonomy in a family of quantum maps under a rank-1 perturbation $U(\lambda) = U_0 e^{-i\lambda|v\rangle\langle v|}$.

Two Bose particles

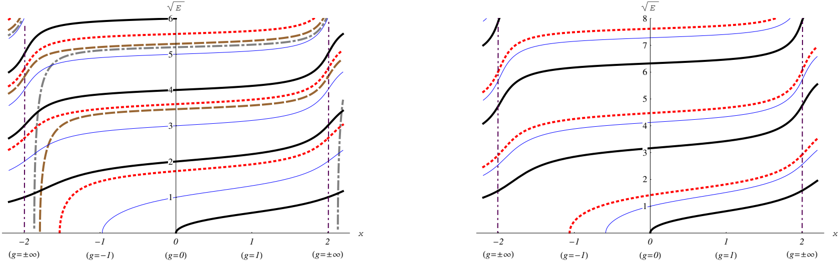
N. Yonezawa, AT and T. Cheon, PRA **87**, (2013).



Parametric evolution of eigenenergies of the two Bose particles in a ring (two-body Lieb-Liniger model), with respect to the coupling strength g. The cycle $g = 0 \to \infty / -\infty \to 0$ induce the exotic quantum holonomy.

Lieb-Liniger model (many Bose particles)

N. Yonezawa, AT and T. Cheon, PRA **87**, (2013).

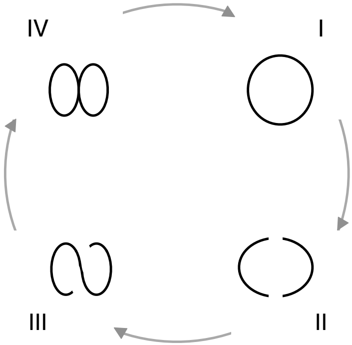


Three- (left) and four- (right) body Lieb-Liniger models.

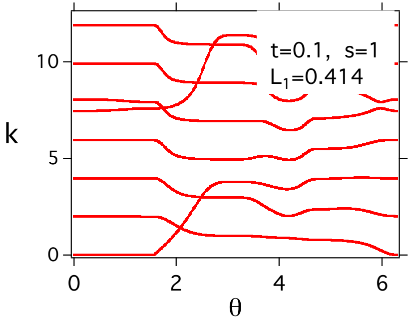
The cycle $g = 0 \to \infty / -\infty \to 0$ can be realized experimentally using confinement induced resonance (Olshanii, PRL **81** (1998), Haller *et al.*, Science **325** (2009)).

In quantum graphs

T. Cheon, AT and O. Turek, Acta Polytech. **53** (2013).



A cycle of quantum graph.



Parametric evolution of $\sqrt{2E}$.

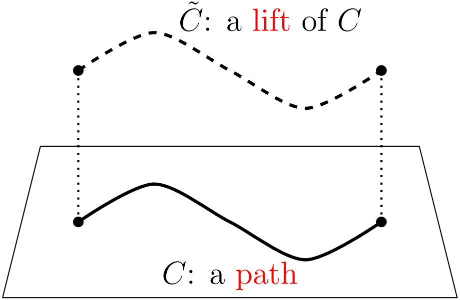
Other examples

- Quantum graphs/Generalized contact potentials (I. Tsutsui, T. Fülop and T. Cheon 2000; I. Tsutsui, T. Fülop and T. Cheon 2001; S. Ohya, Ann. Phys. **331** (2013); S. Ohya, Ann. Phys. **351** (2014))
- Non-Abelian extension (T. Cheon and AT 2009)
- Nonadiabatic example in time-dependent Aharonov-Bohm ring (AT and T. Cheon 2010)
- Accelerating adiabatic quantum computation (AT and K. Nemoto 2010)
- Hierarchical many-qubit systems (AT, S. W. Kim and T. Cheon 2011; AT, T. Cheon and S. W. Kim 2012)
- Autonomous Hamiltonians (T. Cheon, AT and S. W. Kim, 2009)
- Another good example?

Theoretical works

- Generalized Fujikawa formalism (T. Cheon and AT 2009; AT and T. Cheon 2009)
  ...the eigenspace anholonomy and the off-diagonal geometric phase factors (Manini and Pistolesi, PRL **85** (2000)) are *entangled*
- Exotic quantum holonomy as an encirclement of non-Hermitian degeneracy points by Hermitian Hamiltonian/unitary Floquet operators. (S. W. Kim, T. Cheon and AT 2010; AT, N. Yonezawa and T. Cheon 2013; AT, S. W. Kim and T. Cheon 2014)
  ...requires an *analytic continuation* of parameters
- Abelian gerbes in adiabatic Floquet theory (Viennot, JPA **42** (2009))
  ...applicable only to periodically driven systems
- Another good theory?
  ...the main subject of the next section

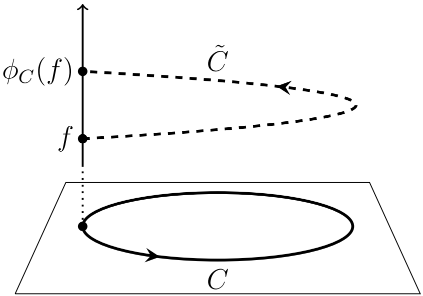
What is missing in theory — lifting structure



- The adiabatic time evolution along C induces $\tilde{C}$ (cf. Simon 1983).
- The parameterization of path by *quantum dynamical variable* completes a geometrical picture, and offers a nonadiabatic extension (cf. Aharonov and Anandan 1987).

What is available from the lifting structure

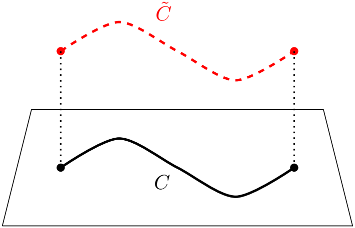
The lift $\tilde{C}$ connects f with $\phi_C(f)$, for a given a *cycle (closed path)* C.

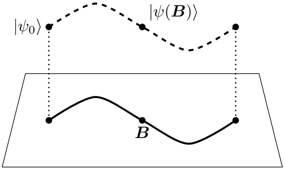


- How C determines ϕ_C?

cf. As for the phase holonomy, ϕ_C is equivalent to the *geometric phase factor*, and is an element of the holonomy group (Simon 1983).

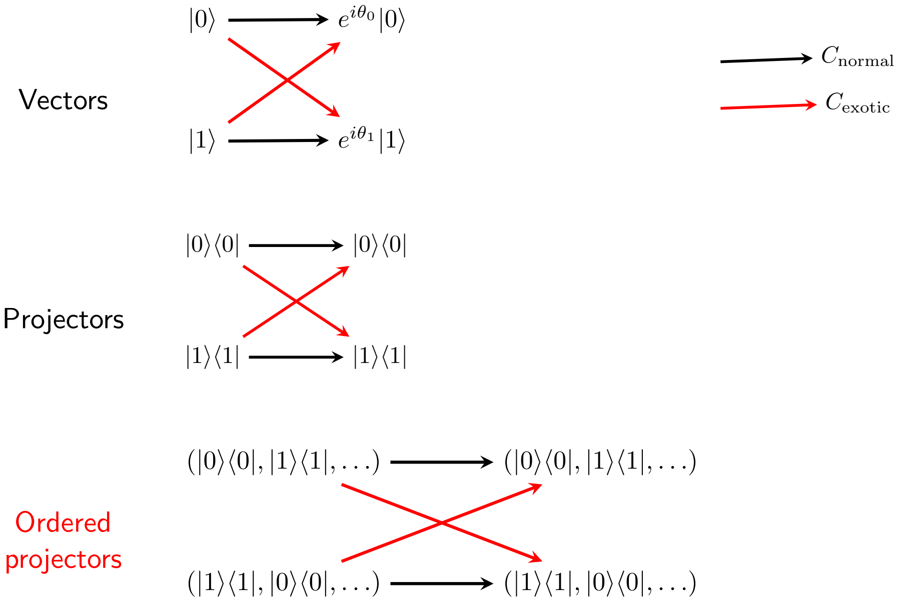
Problem 1: Where to lift a path C?





cf. As for the geometric phase, the lift is a trajectory of state vector, which obeys the adiabatic Schrödinger equation.

Changes in eigenobjects by cycles

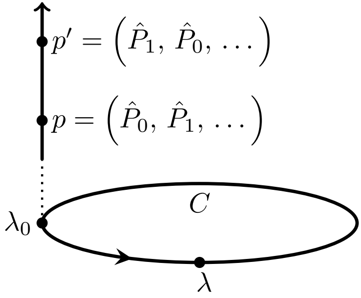


An ordered set of mutually orthogonal projectors

Let P_n denote the n-th eigenprojector for a given value of the adiabatic parameter, say λ_0. We define an ordered set of mutually orthogonal projectors as

$$p \equiv (P_0, P_1, \dots).$$

The value of p is "quantized", since the order of P_n is arbitrary for λ_0.



Hence p-space can be regarded as a fiber with a *discrete* structure group (cf. as for the geometric phase, *continuous* in general).

Lifting a cycle C to p-space

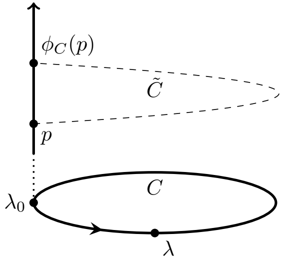
The lift $\tilde{C}$ connects

$$p = (P_0, P_1, \dots)$$

with $\phi_C(p)$, for example,

$$\phi_C(p) = (P_1, P_0, \dots)$$

i.e., ϕ_C describes the permutation of eigenspaces induced by the adiabatic time evolution along C.

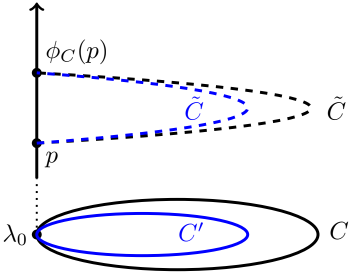


How C determines ϕ_C? — homotopy classification of C

The theory of covering map (homotopy lifting property) tells us

$$\phi_C = \phi_{C'} \quad \text{if } C \text{ and } C' \text{ are homotopic.}$$

Hence we denote $\phi_{[C]}$ instead of ϕ_C, where $[C]$ is the class of cycles that are homotopic to C.



The set of all possible $[C]$ in M is the first fundamental group $\pi_1(M)$, which plays the central role to understand $\phi_{[C]}$'s.

Example: Kicked spin-$\frac{1}{2}$

A time-periodic kick is applied to a spin-$\frac{1}{2}$ under a static magnetic field $\boldsymbol{B}$:

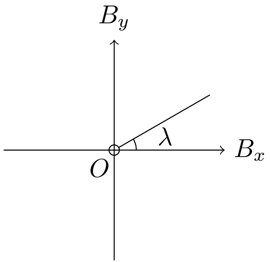
$$H(t) \equiv \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{\sigma} + \lambda\frac{1-\sigma_z}{2}\sum_{m=-\infty}^{\infty}\delta(t-m),$$

whose Floquet operator is

$$U \equiv e^{-i\lambda\frac{1-\sigma_z}{2}}e^{-\frac{i}{2}\boldsymbol{B}\cdot\boldsymbol{\sigma}}.$$

We choose $\boldsymbol{B} = (B_x, B_y, 0)$ and $\lambda = \tan^{-1}(B_y/B_x)$, which ensures the single-valuedness of U.

From the parameter space M (a part of $\boldsymbol{B}$-plane), the degenerate point O is excluded.



Kicked spin-$\frac{1}{2}$: parametric evolution of eigenprojector

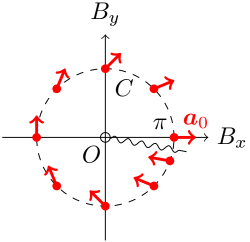
Let $|n\rangle$ be the n-th eigenvector of U, i.e., $U|n\rangle = z_n|n\rangle$ $(n = 0, 1)$.

Eigenprojectors can be specified by a normalized "Bloch vector" $\boldsymbol{a}(= \langle 0|\boldsymbol{\sigma}|0\rangle)$ as

$$|0\rangle\langle 0| = P(\boldsymbol{a}), \quad |1\rangle\langle 1| = P(-\boldsymbol{a}),$$

where

$$P(\boldsymbol{a}) = \frac{1 + \boldsymbol{\sigma}\cdot\boldsymbol{a}}{2}.$$



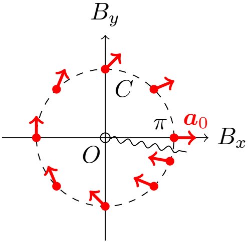
$\boldsymbol{a}$ is multiple-valued due to the eigenspace anholonomy.

Kicked spin-$\frac{1}{2}$: an analysis of anholonomy

As for two level systems, the ordered projector p is equivalent to the Bloch vector $\boldsymbol{a}$ since

$$p = (\ P(\boldsymbol{a}), P(-\boldsymbol{a})\ )$$

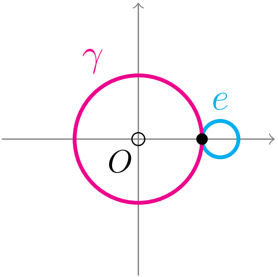
holds, where $P(\boldsymbol{a}) = (1 + \boldsymbol{\sigma}\cdot\boldsymbol{a})/2$.



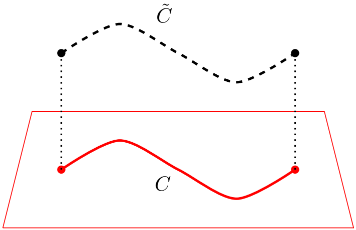
Since M is 2-dim plane excluded the origin, its fundamental group is $\pi_1(M) = \{[e], [\gamma], [\gamma^2], \ldots\}$. Because of $\phi_{[\gamma^2]} = \phi_{[e]}$, there are only two kinds of $\phi_{[C]}$, i.e.,

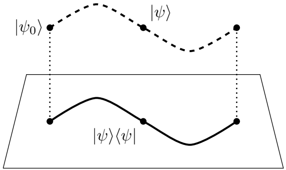
$$\left\{\phi_{[C]}\right\}_{[C]\in\pi_1(M)} \simeq \mathbb{Z}_2,$$

which corresponds to the identical and cyclic permutations of two items.



Problem 2: How C is parameterized by dynamical variables?





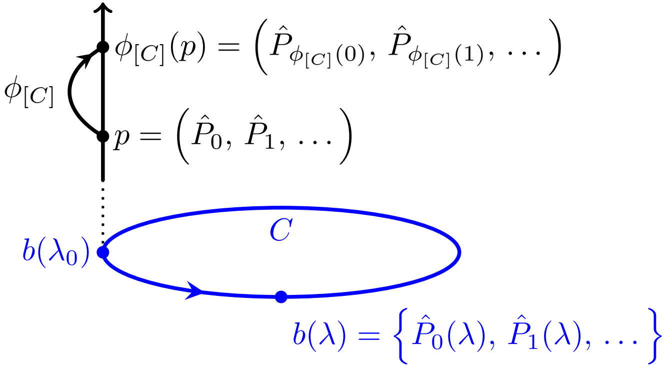
cf. For the geometric phase, the space of projectors (the projective Hilbert space) may parameterize cycles (Aharonov and Anandan 1987).

Cycles in terms of a dynamical variable b

Let b denote the set of mutually orthogonal eigenprojectors, i.e.,

$$b \equiv \{P_0, P_1, \dots\},$$

where the order of P_n's are disregarded.



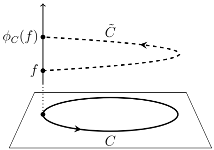
Topological formulation: a summary

Behind the exotic quantum holonomy, we find a covering structure (a fiber bundle with discrete structure group) consists of $\mathcal{P}$ and $\mathcal{B}$.

1. $\mathcal{P}$ consists of $(P_0, P_1, \dots)$.
2. $\mathcal{B}$ may be a c-number parameter space, or, may consist of $\{P_0, P_1, \dots\}$. The latter offers the parameterization of C by dynamical variables.
3. $\phi_{[C]}$ (the permutation of eigenspaces induced by C) and $\pi_1(\mathcal{B})$ has 1:1 correspondence, i.e.

$$\{\phi_{[C]}\}_{[C]\in\pi_1(\mathcal{B})} \simeq \pi_1(\mathcal{B}),$$

   when $\pi_1(\mathcal{P})$ is simply connected (i.e., $\pi_1(\mathcal{P})$ has only a single element).



4. Rigorously, we have a formula $\{\phi_{[C]}\}_{[C]\in\pi_1(\mathcal{B})} \simeq \pi_1(\mathcal{B})/\pi_*\{\pi_1(\mathcal{P})\}$, where a projector $\pi : \mathcal{P} \to \mathcal{B}$ is called a covering map.

Application: Classify all two level Floquet systems

U: a Floquet operator (time evolution operator for the unit time interval) of a periodically driven two level system

A spectral decomposition of U:

$$U = z_+ P(\boldsymbol{a}) + z_- P(-\boldsymbol{a})$$

where $z_\pm$ are the unimodular eigenvalues, and $P(\boldsymbol{a})$ is a projection operator parameterized by a normalized vector $\boldsymbol{a}$:

$$P(\boldsymbol{a}) = \frac{1 + \boldsymbol{a} \cdot \boldsymbol{\sigma}}{2}.$$

Note that $P(\boldsymbol{a})P(-\boldsymbol{a}) = 0$ holds (orthogonality).

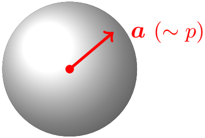
$\pm\boldsymbol{a}$ are the Bloch vectors of eigenstates $P(\pm\boldsymbol{a})$, respectively.

Parameterization of p by "Bloch vector" $\boldsymbol{a}$

In two-level systems, $p \equiv (P_0, P_1)$ can be parameterized by a normalized Bloch vector $\boldsymbol{a}$ as

$$p = (P(\boldsymbol{a}), P(-\boldsymbol{a}))\,.$$

Hence we identify $\mathcal{P}$ with a sphere S^2.



Parameterization of b by the director $\boldsymbol{n}$

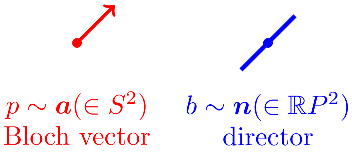
In two-level systems, the parameterization of $b \equiv \{P_0, P_1\}$ by the normalized Bloch vector $\boldsymbol{a}$ is redundant

$$b = \{P(\boldsymbol{a}), P(-\boldsymbol{a})\},$$

because the order of the elements in b makes no distinction, i.e.,

$$b = \{P(+\boldsymbol{a}), P(-\boldsymbol{a})\} = \{P(-\boldsymbol{a}), P(+\boldsymbol{a})\}$$

holds. Here we identify b with the director (headless vector) $\boldsymbol{n}$, which is a point in the real projective plane $\mathbb{R}P^2$. Hence $\mathcal{B} \simeq \mathbb{R}P^2$.



$\phi_{[C]}$ and the fundamental group $\pi_1(\mathcal{B})$

For two level systems, because of $\pi_1(\mathcal{P}) = 1$ ($\because \mathcal{P} \simeq S^2$),

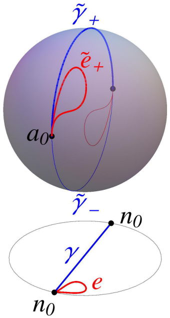
$$\{\phi_{[C]}\}_{[C]\in\pi_1(\mathcal{B})} \simeq \pi_1(\mathcal{B})$$

holds, i.e., $\pi_1(\mathcal{B})$ governs the eigenspace anholonomy ($\mathcal{B} = \mathbb{R}P^2$).

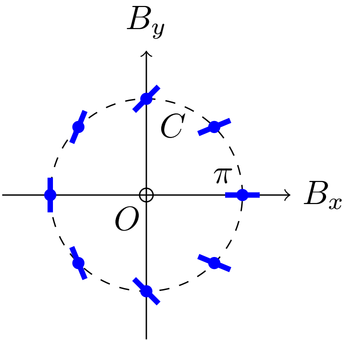
Each element of $\pi_1(\mathbb{R}P^2) = \{[e], [\gamma]\}$ has a $1:1$ correspondence with the permutation σ of the eigenspaces.

- $[e] \leftrightarrow$ the identical permutation ($\sim$ the absence of the anholonomy)
- $[\gamma] \leftrightarrow$ the cyclic permutation ($\sim$ the presence the anholonomy)

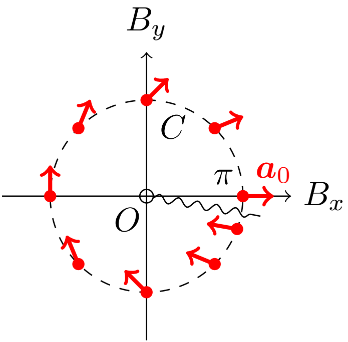
Hence $\{\phi_{[C]}\}_{[C]\in\pi_1(\mathcal{B})} \simeq \mathbb{Z}_2$ also holds here.



Kicked spin-$\frac{1}{2}$: disclination (line defect) of $\boldsymbol{n}$



$\boldsymbol{n}$ exhibits disclination (line defect) in (B_x, B_y)-plane.

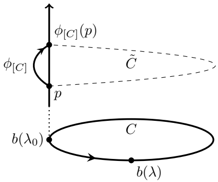


Because of the disclination, the trajectory of $\boldsymbol{a}$ along C is open.

Summary

The base and superstructure in the exotic quantum holonomy are identified to establish the topological formulation.

Conventional	Exotic
$\vert\psi\rangle$	$f = (\vert 0\rangle, \vert 1\rangle, \dots)$
Geometric Phase	off-diagonal GP
$\hat{P}\ (= \vert\psi\rangle\langle\psi\vert)$	$p = \left(\hat{P}_0, \hat{P}_1, \dots\right)$ **(*ordered* projectors)**
	eigenspace anholonomy
	$b = \left\{\hat{P}_0, \hat{P}_1, \dots\right\}$ **(the order is *ignored*)**



The homotopic classification of cycles (closed paths) play the central role in the exotic quantum holonomy.

Ref. AT and T. Cheon, `arXiv:1402.1634` (Phys. Lett. A, **379** (2015) p.1693).