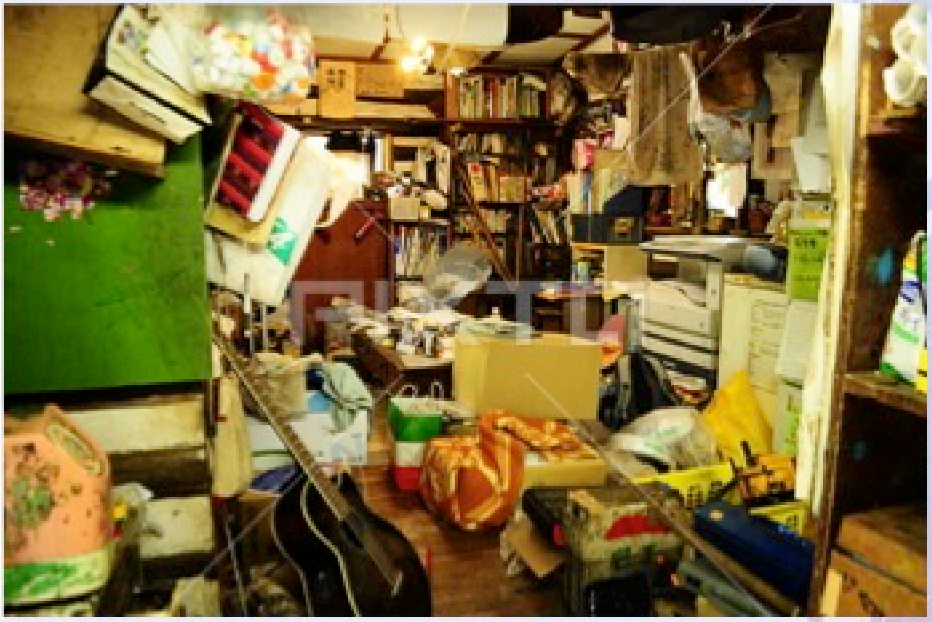
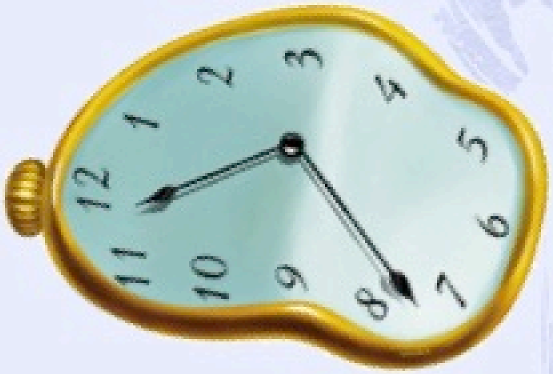
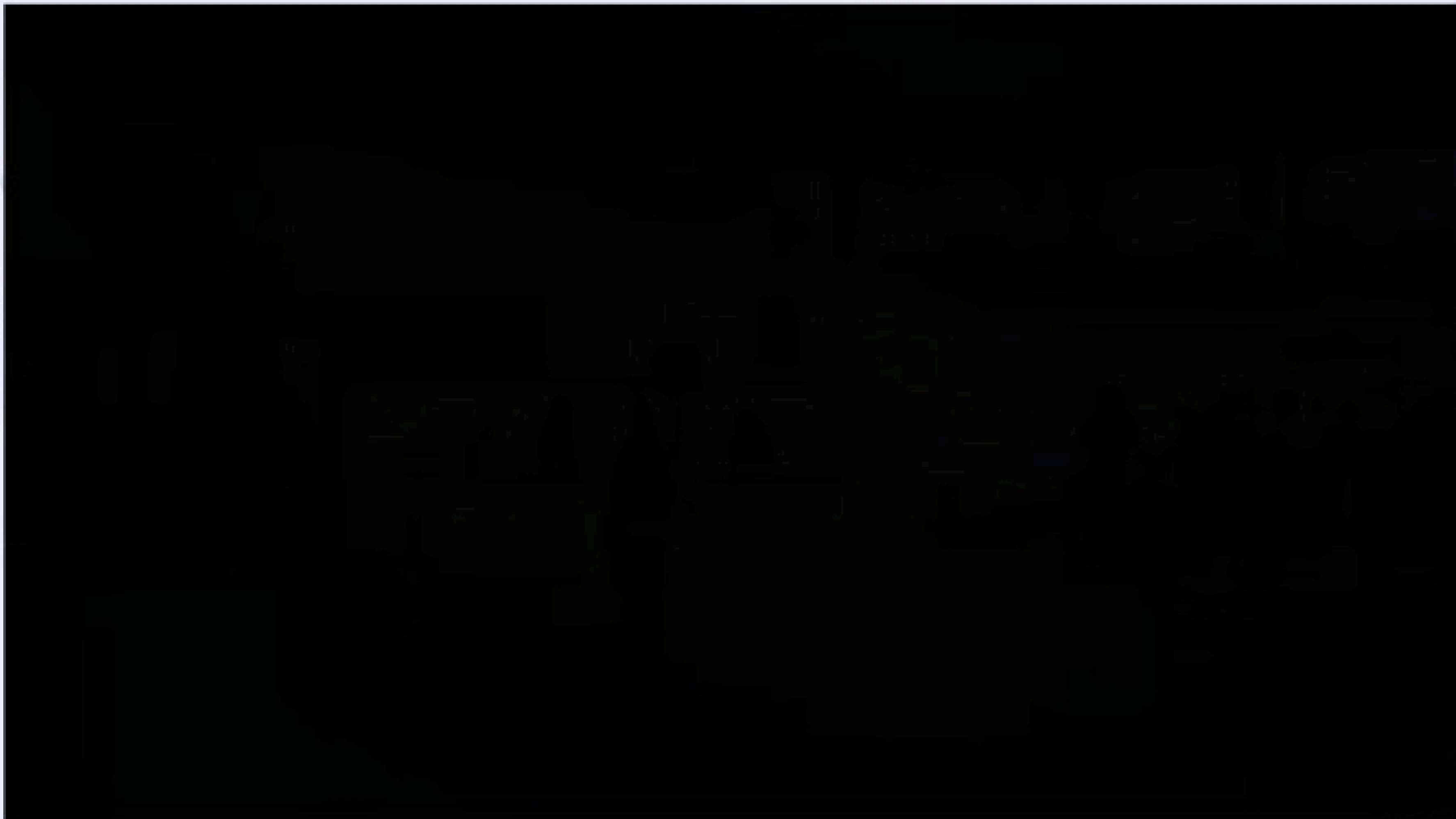
Arrow of Time



Arrow of Time



Arrow of Time



Reverse Brothers (in Japanese) http://nori510.com/

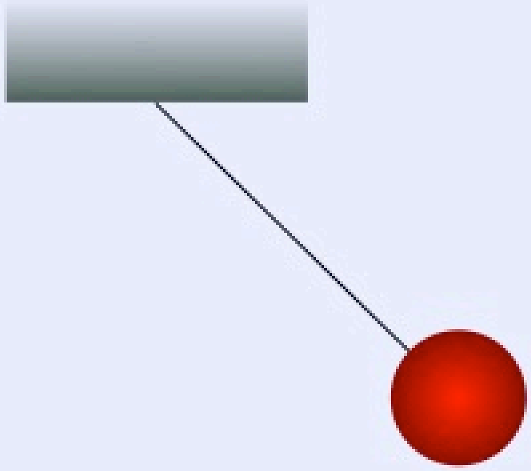
Newton's Equation of Motion

$$m\frac{d^2x}{dt^2} = F$$

Symmetric with respect to the time reversal $t \to -t$

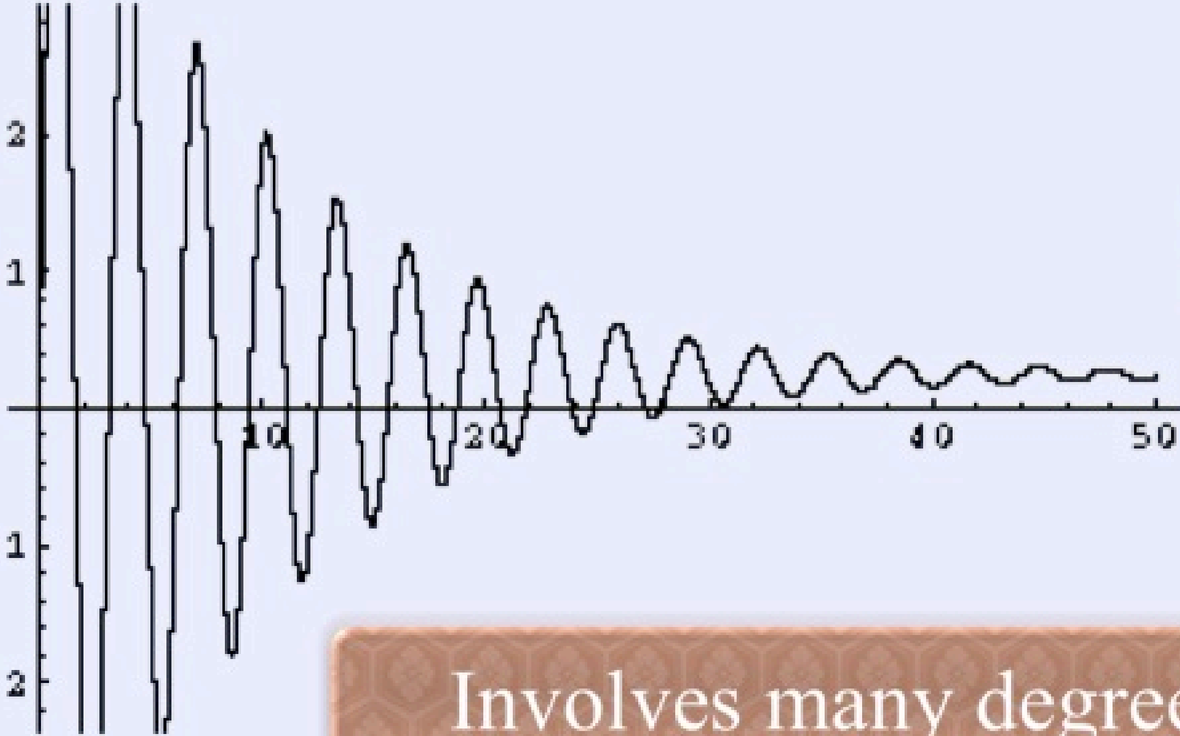
Time-Reversal Symmetry

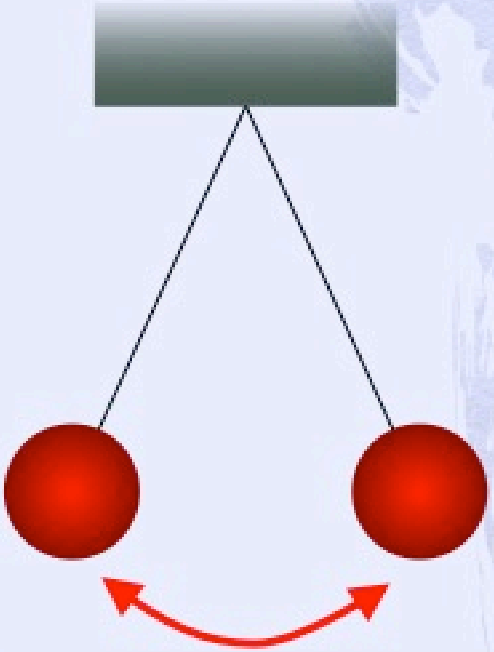
No Arrow of Time



Time-Reversal Symmetry

Arrow of Time



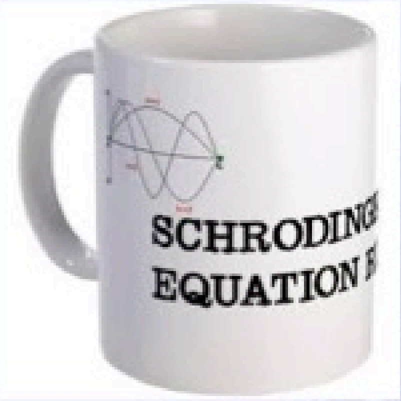


Involves many degrees of freedom

Broken Time-Reversal Symmetry

Schrödinger's Wave Equation

$$i\frac{d}{dt}\psi = H\psi$$

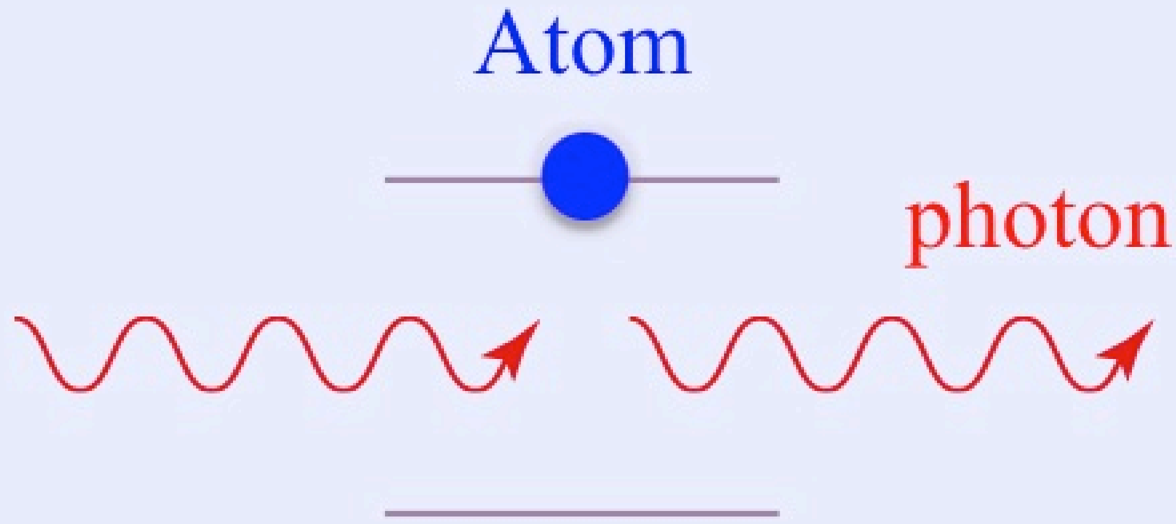


Symmetric with respect to
the time reversal $t \to -t$
and $i \to -i$

Time-Reversal Symmetry

Arrow of Time

Einstein coefficients



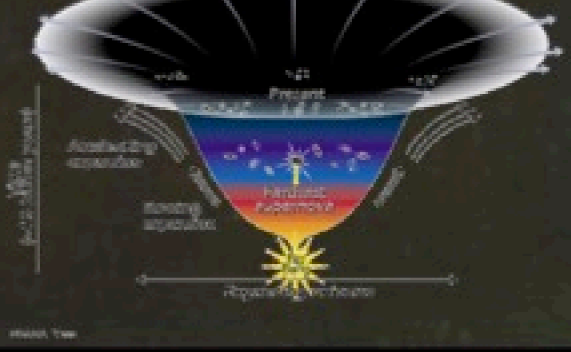
Absorption

Spontaneous Emission

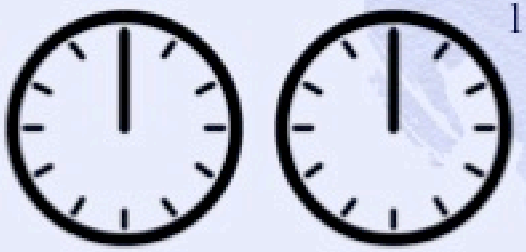
Broken Time-Reversal Symmetry

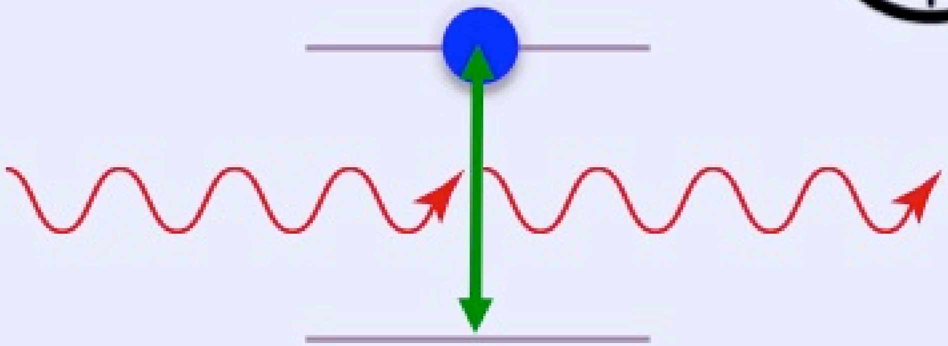
Arrows of Time

- Dynamical Arrow of Time
- Thermodynamical Arrow of Time
- Psychological Arrow of Time
- Cosmological Arrow of Time



Today's Message



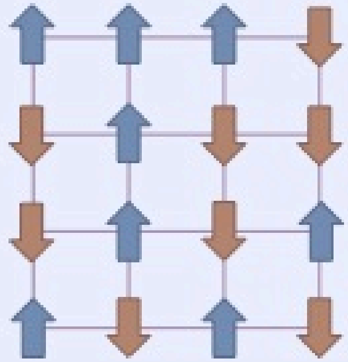


- The excited state decays in the time evolution from an initial condition.
- The excited state grows in the time evolution towards a terminal condition.

Peierls, Feynman, Hawking

Spontaneous Symmetry Breaking

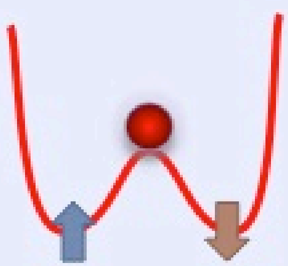
$T > T_{\mathrm{c}}$

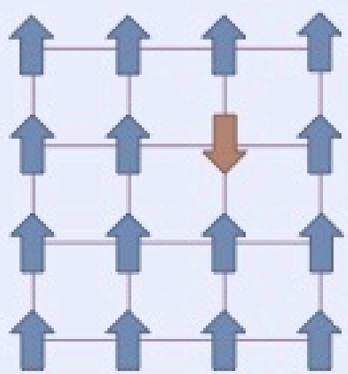


Ising model

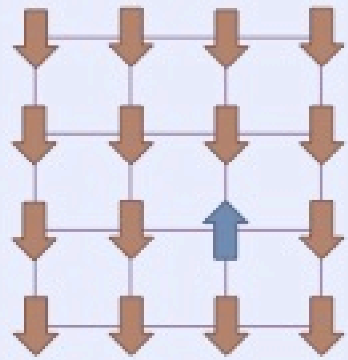
$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

$T < T_{\mathrm{c}}$





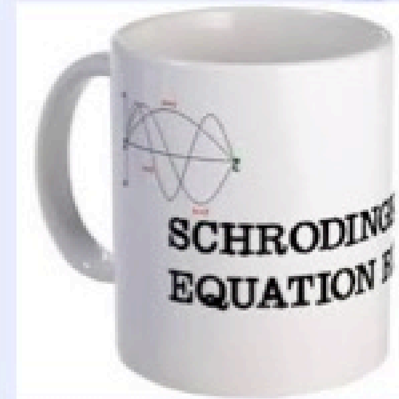
or

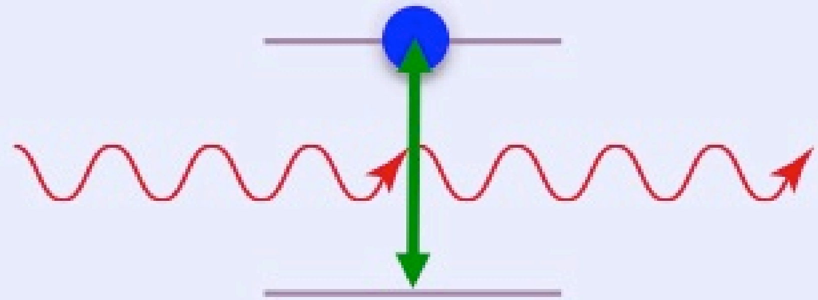


Intrinsic reason $N \to \infty$	**Extrinsic reason**
Symmetry-breaking states	Choice of an external field

Time-reversal symmetry breaking

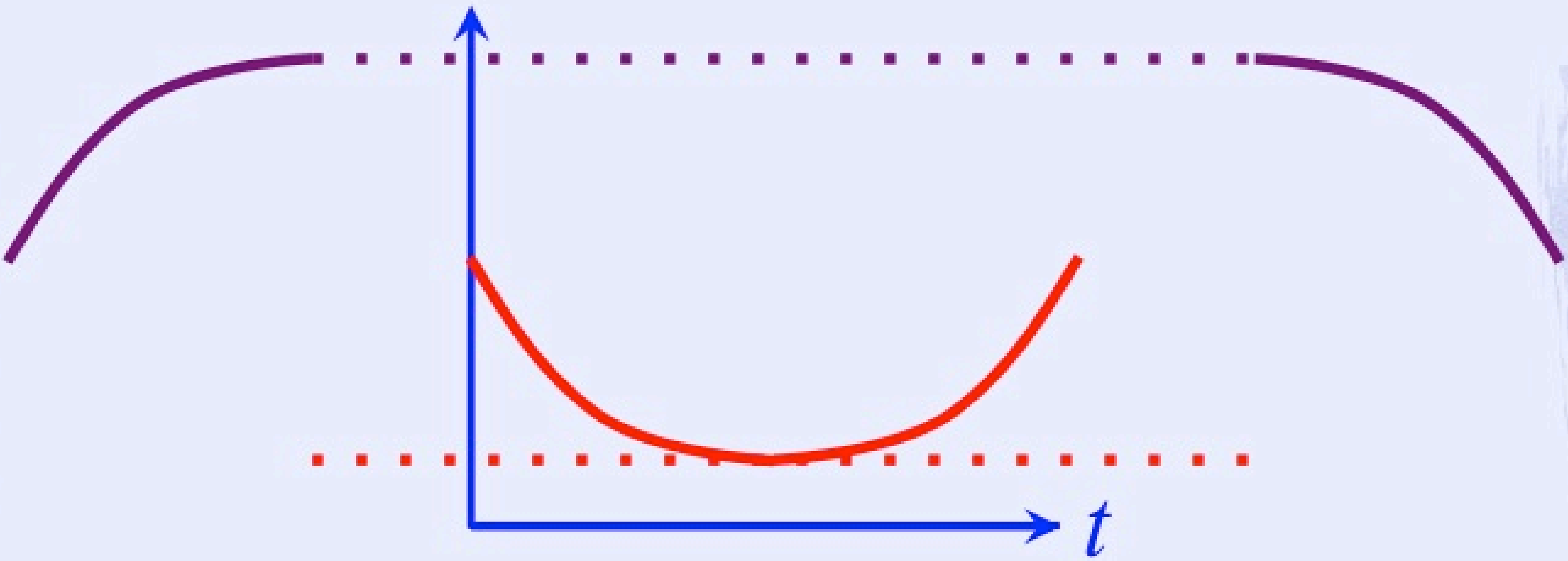
$$i\frac{d}{dt}\psi = H\psi$$





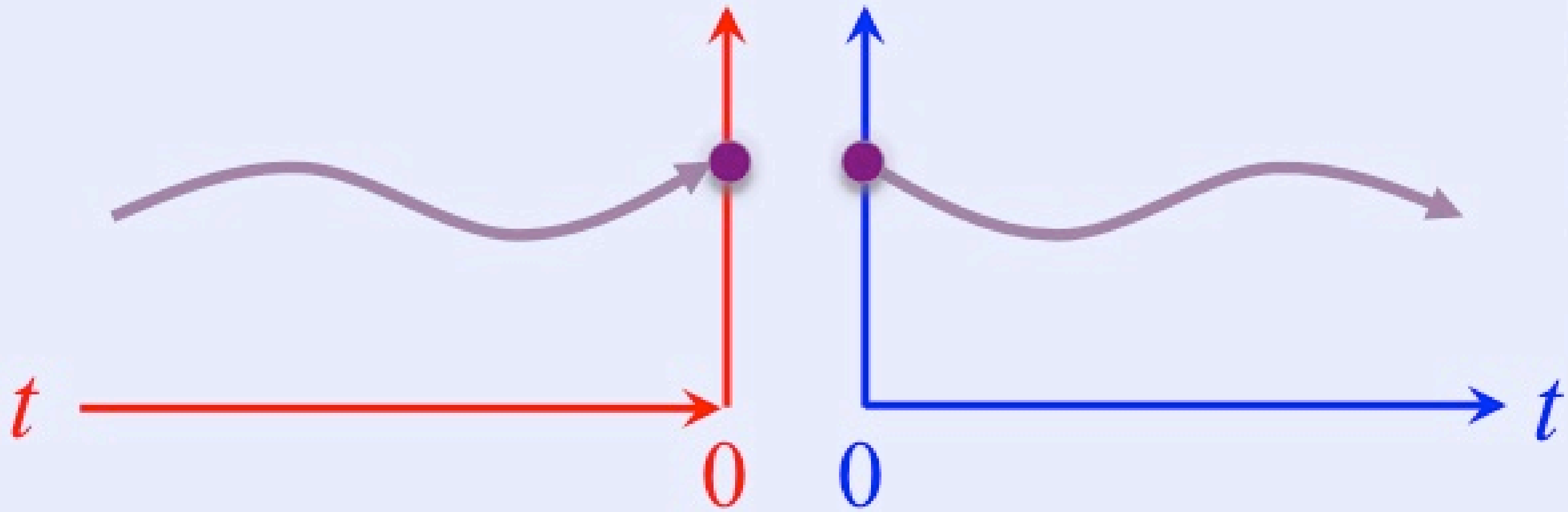
Intrinsic reason $N \to \infty$	**Extrinsic reason**
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

Recurrence Time



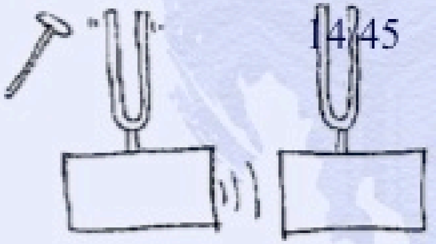
Intrinsic reason $N \to \infty$	**Extrinsic reason**
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

The initial condition is easier?

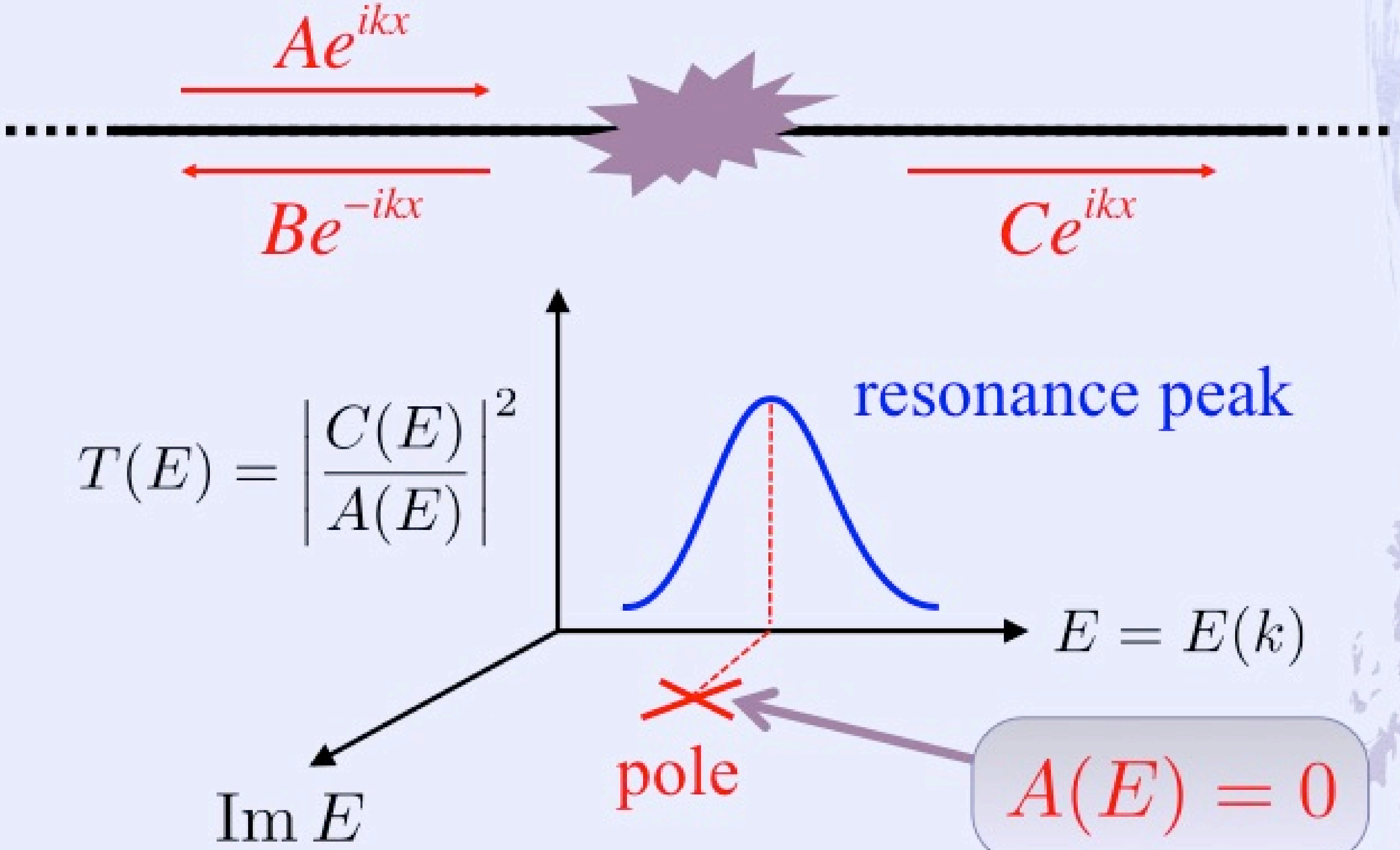


Intrinsic reason $N \to \infty$	**Extrinsic reason**
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

Definition of resonance



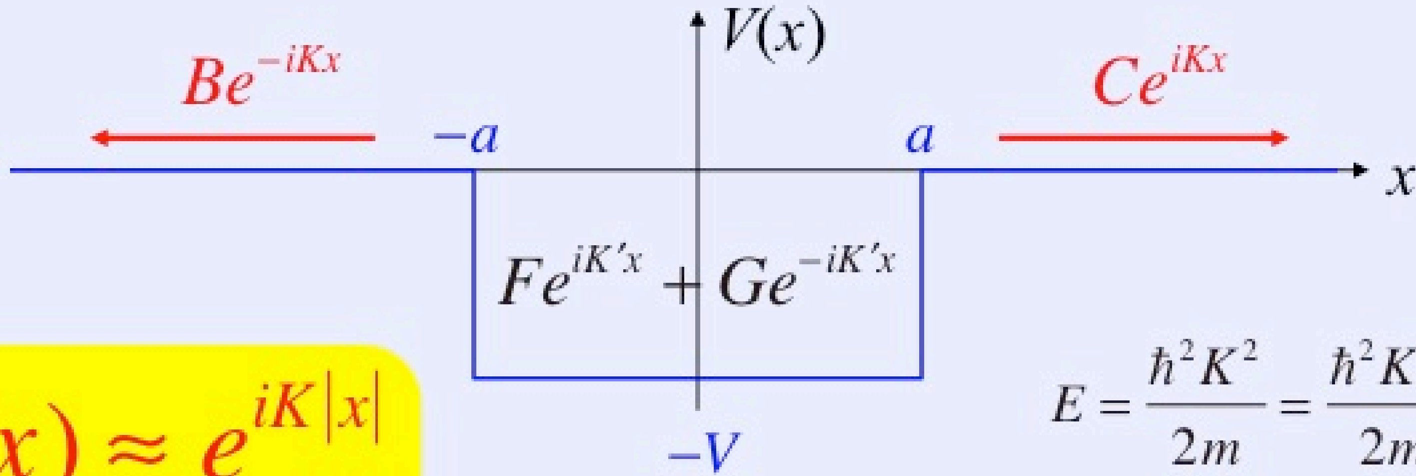
Resonance: Pole of Transmission Probability



Definition of resonance

Siegert condition (1939)
Resonance: Eigenstate with outgoing waves only.

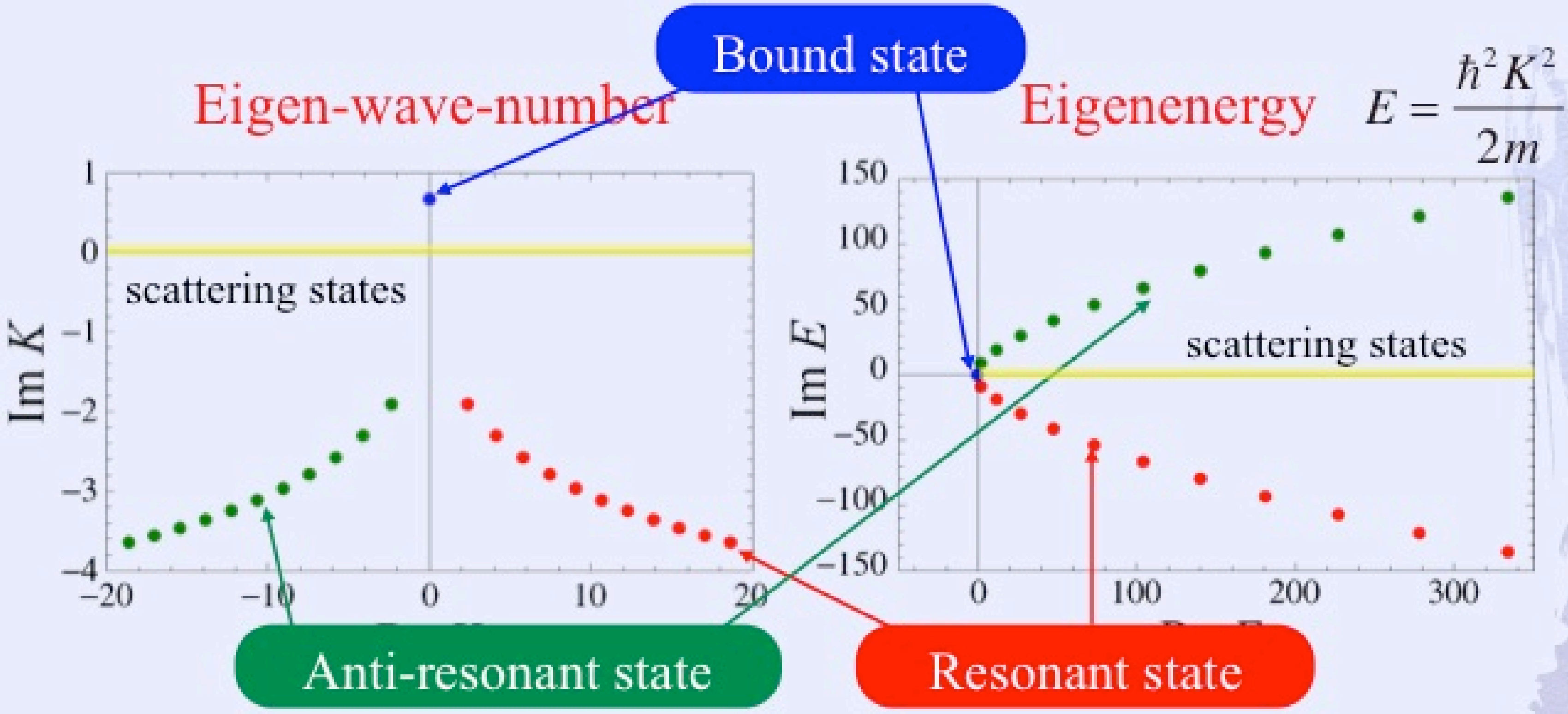
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)$$



$$\psi(x)\approx e^{iK|x|}$$

$$E=\frac{\hbar^2K^2}{2m}=\frac{\hbar^2K'^2}{2m}-V$$

Definition of resonance



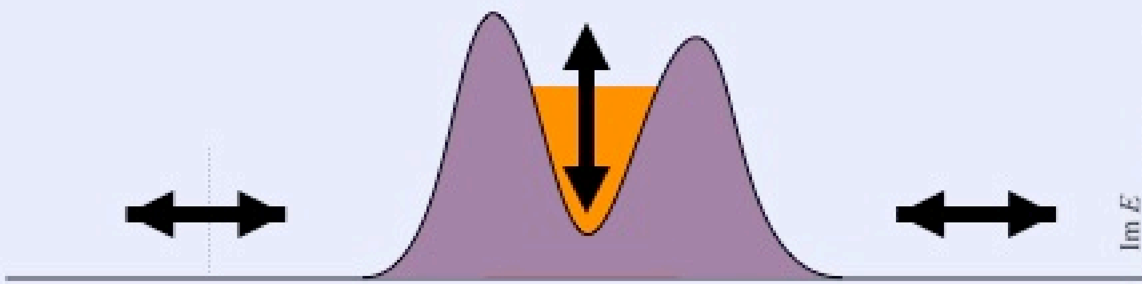
$$\psi(x) \approx e^{iK|x|}$$

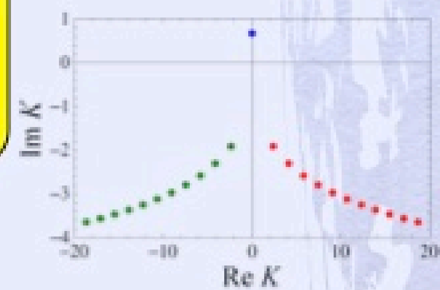
$$\begin{cases} \operatorname{Re} K_n \gtrless 0 \Leftrightarrow \operatorname{Im} E_n \lessgtr 0 \\ \operatorname{Im} K_n < 0 \end{cases}$$

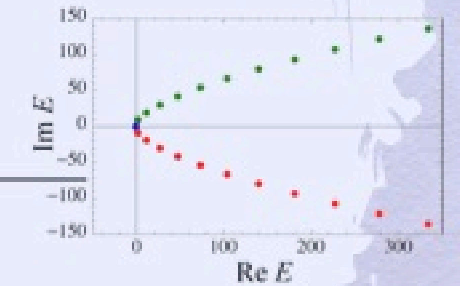
Resonant state as a stationary eigenstate

N. Hatano, K. Sasada, H. Nakamura and T. Petrosky, Prog. Theor. Phys. **119** (2008) 187

$$\langle x | \Psi_n(t) \rangle \approx e^{iK_n |x| - iE_n t}$$







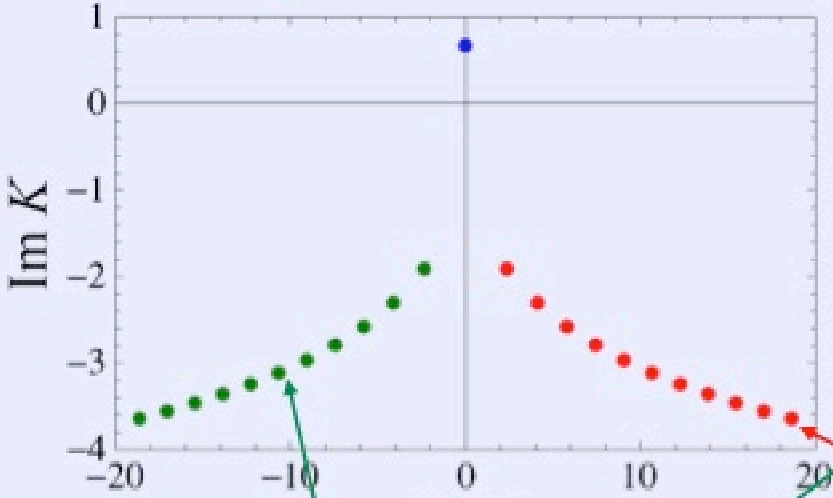
$$\mathrm{Im}\, E_n \gtrless 0 \Leftrightarrow \mathrm{Re}\, K_n \gtrless 0$$

"Resonant state" as an eigenstate

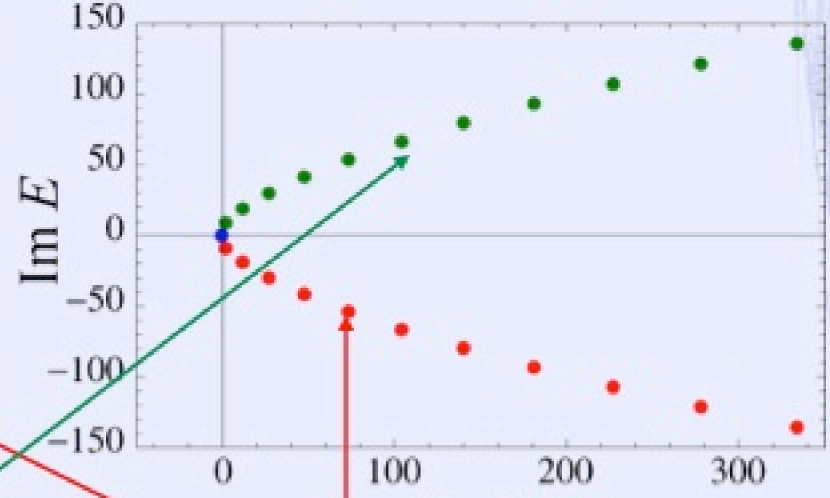
Time-Reversal Symmetry

N. Hatano, K. Sasada, H. Nakamura and T. Petrosky, Prog. Theor. Phys. **119** (2008) 187

Eigen-wave-number



Eigenenergy



Anti-resonant state

Growing solution

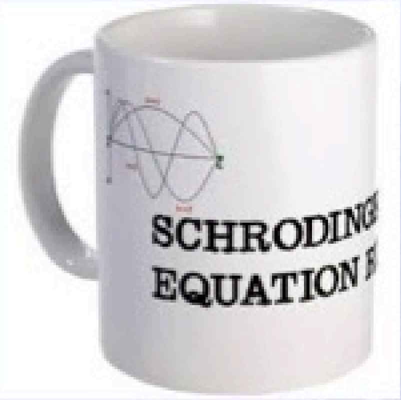
Resonant state

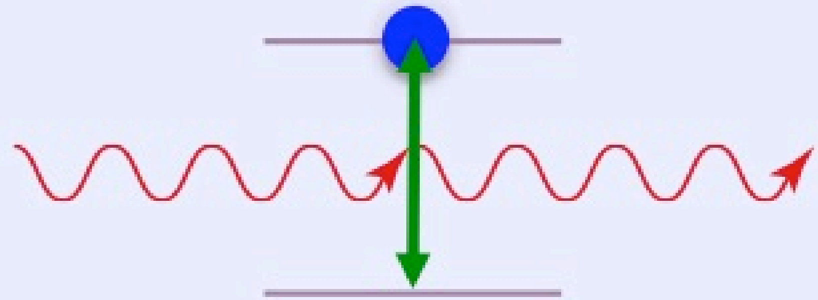
Decaying solultion

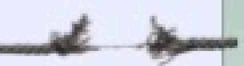
Each resonant or anti-resonant state breaks the time-reversal symmetry spontaneously!

Schrödinger's Wave Equation

$$i\frac{d}{dt}\psi = H\psi$$

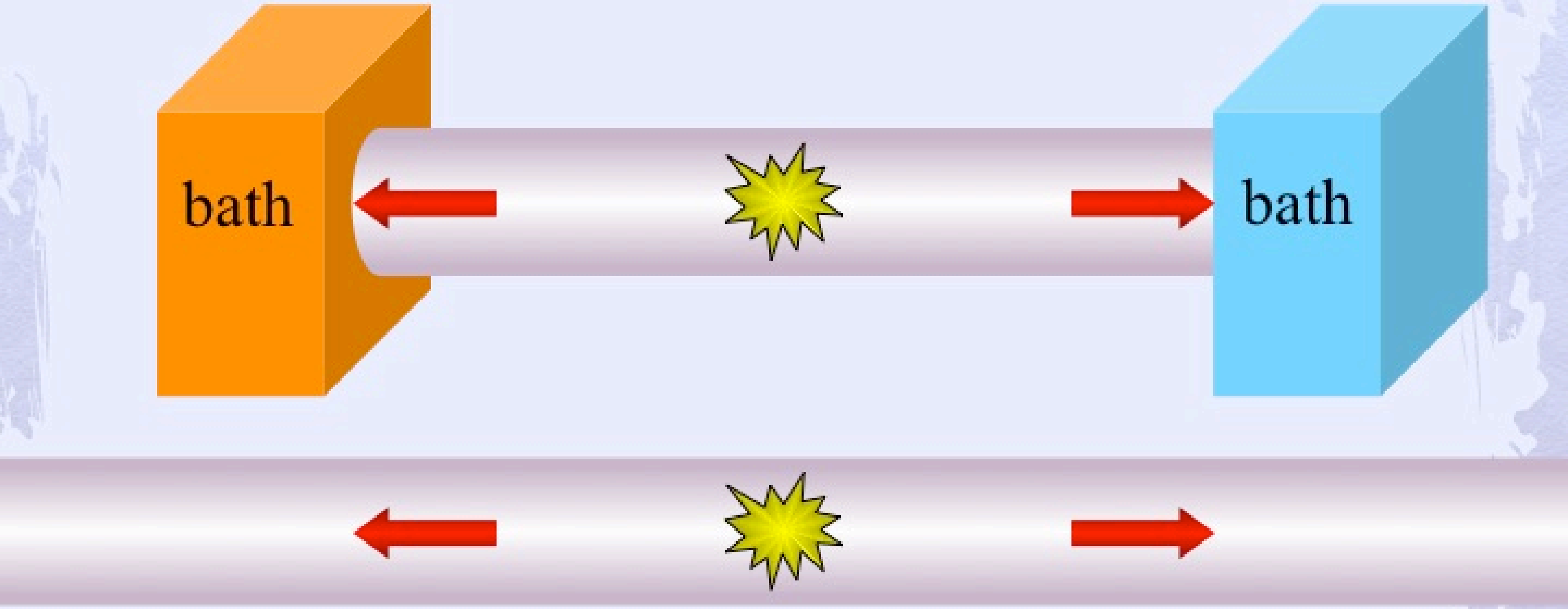






Intrinsic reason $N \to \infty$	**Extrinsic reason**
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

Landauer formula

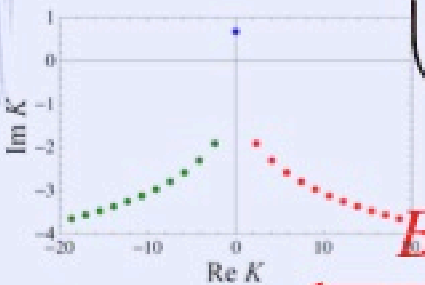


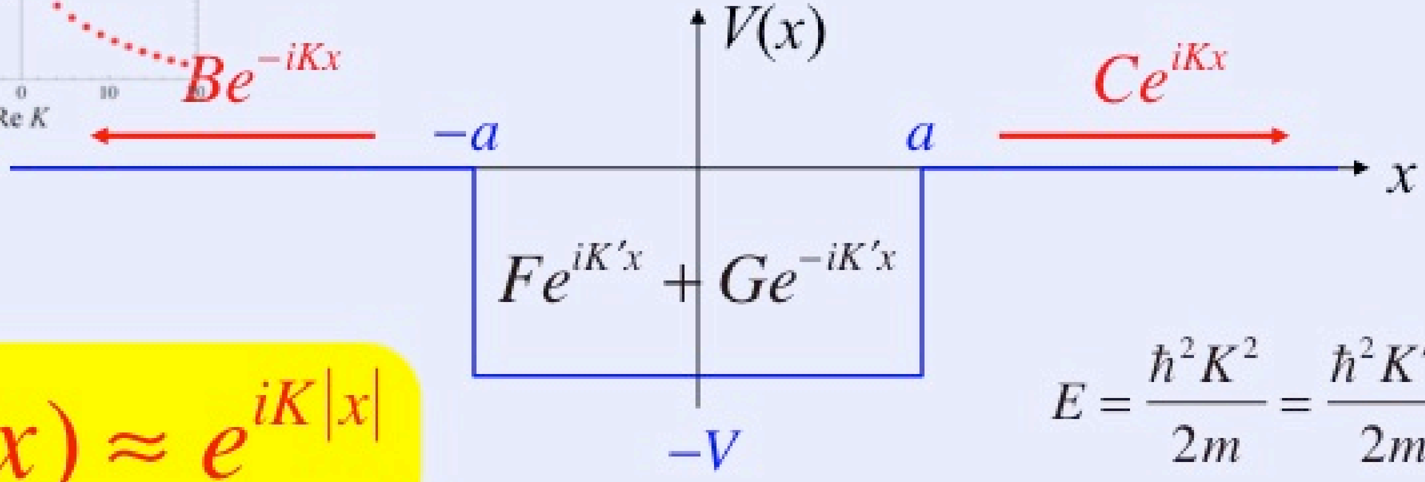
Once the particles go out of the central region, they never come back.

Question

Why can complex eigenvalues appear out of a Hermitian Hamiltonian?

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)$$





$$\psi(x)\approx e^{iK|x|}$$

$$E=\frac{\hbar^2K^2}{2m}=\frac{\hbar^2K'^2}{2m}-V$$

Non-Hermiticity of open systems

N. Hatano, K. Sasada, H. Nakamura and T. Petrosky, Prog. Theor. Phys. **119** (2008) 187

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \quad \text{where} \quad \operatorname{Im} V(x) \equiv 0$$

$\Omega = [-L, L]$

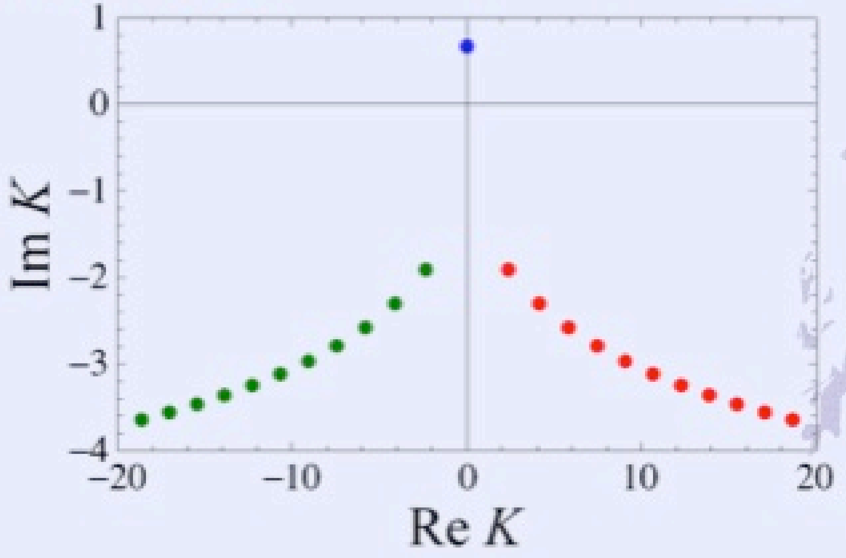
$$\langle \psi | \hat{p}^2 | \psi \rangle_\Omega = -\hbar^2 \int_{-L}^{L} \psi(x)^* \psi''(x) dx = -\hbar^2 \left[\psi(x)^* \psi'(x) \right]_{x=-L}^{L} + \hbar^2 \int_{-L}^{L} \psi'(x)^* \psi'(x) dx$$

$$-\big) \ \langle \psi | \hat{p}^2 | \psi \rangle_\Omega^{\ *} = -\hbar^2 \int_{-L}^{L} \psi(x) \psi''(x)^* dx = -\hbar^2 \left[\psi(x) \psi'(x)^* \right]_{x=-L}^{L} + \hbar^2 \int_{-L}^{L} \psi'(x) \psi'(x)^* dx$$

$$2i \operatorname{Im} \langle \psi | \hat{p}^2 | \psi \rangle_\Omega = -\hbar^2 \left[\psi(x)^* \psi'(x) - \psi(x) \psi'(x)^* \right]_{x=-L}^{L} = -2i\hbar \operatorname{Re} \left[\psi(x)^* \hat{p} \psi(x) \right]_{x=-L}^{L}$$

$$\psi(x) \approx e^{iK|x|}$$

Diverging in space.
→ Outside the standard functional space.



Question

Why can complex eigenvalues appear out of a Hermitian Hamiltonian?

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)$$

The seemingly Hermitian Hamiltonian can be non-Hermitian in a wider functional space.

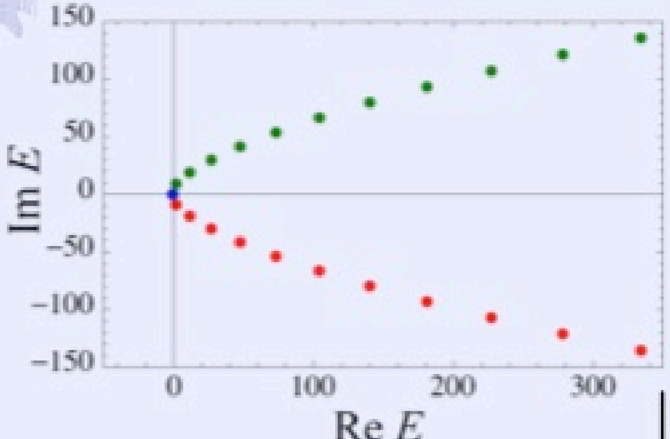
The non-Hermiticity is hidden in the openness of the system.

Question $[H,T]=0$

Don't you have a simultaneous eigenstate of *H* and *T* ?

$$H|\psi\rangle = E|\psi\rangle$$
$$T|\psi\rangle \propto |\psi\rangle$$

$$H(T|\psi\rangle) = T(H|\psi\rangle)$$
$$= TE|\psi\rangle$$
~~$$= E(T|\psi\rangle)$$~~
$$= E^*(T|\psi\rangle)$$



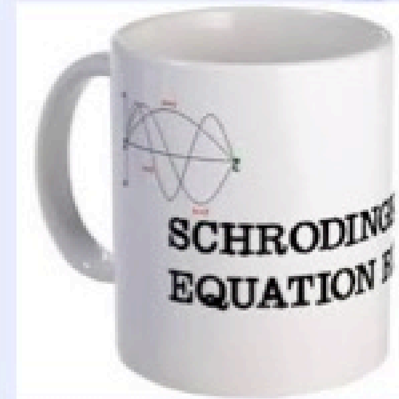
$|\psi\rangle$: a resonant state with E

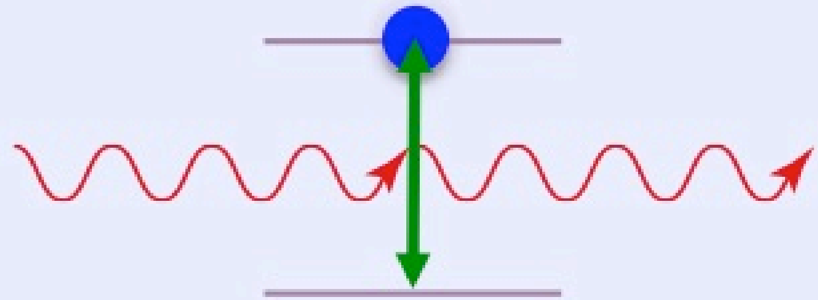
$T|\psi\rangle = |\psi\rangle^*$: an anti-resonant state with E^*


Time flies like
an arrow.
Fruit flies like
a banana.

Time-reversal symmetry breaking

$$i\frac{d}{dt}\psi = H\psi$$





Intrinsic reason $N \to \infty$	Extrinsic reason
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

Time-evolution operator

N. Hatano, G. Ordonez, arXiv:1405.6683v1

$$i\frac{d}{dt}\psi = H\psi$$

$$\Leftrightarrow \quad |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

$t > 0$: decaying solutions (resonant states)

$t < 0$: growing solutions (anti-resonant states)

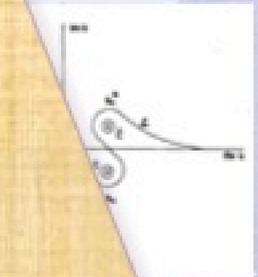
$$\Leftrightarrow \quad e^{-iHt} = \int e^{-iEt}\frac{1}{E-H}dE$$

Time-reversal symmetric expansion

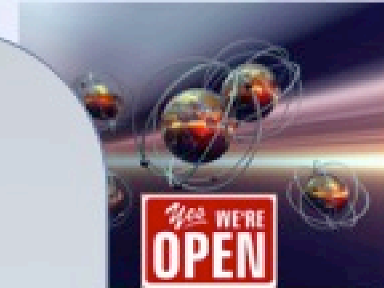
Conventional approach

$$P\frac{1}{E-H}P=\sum_{n=1}^{2N}P|\psi_n\rangle\frac{\lambda\lambda_n}{\lambda-\lambda_n}\langle\tilde{\psi}_n|P$$

Time-reversal symmetric expansion without background integral



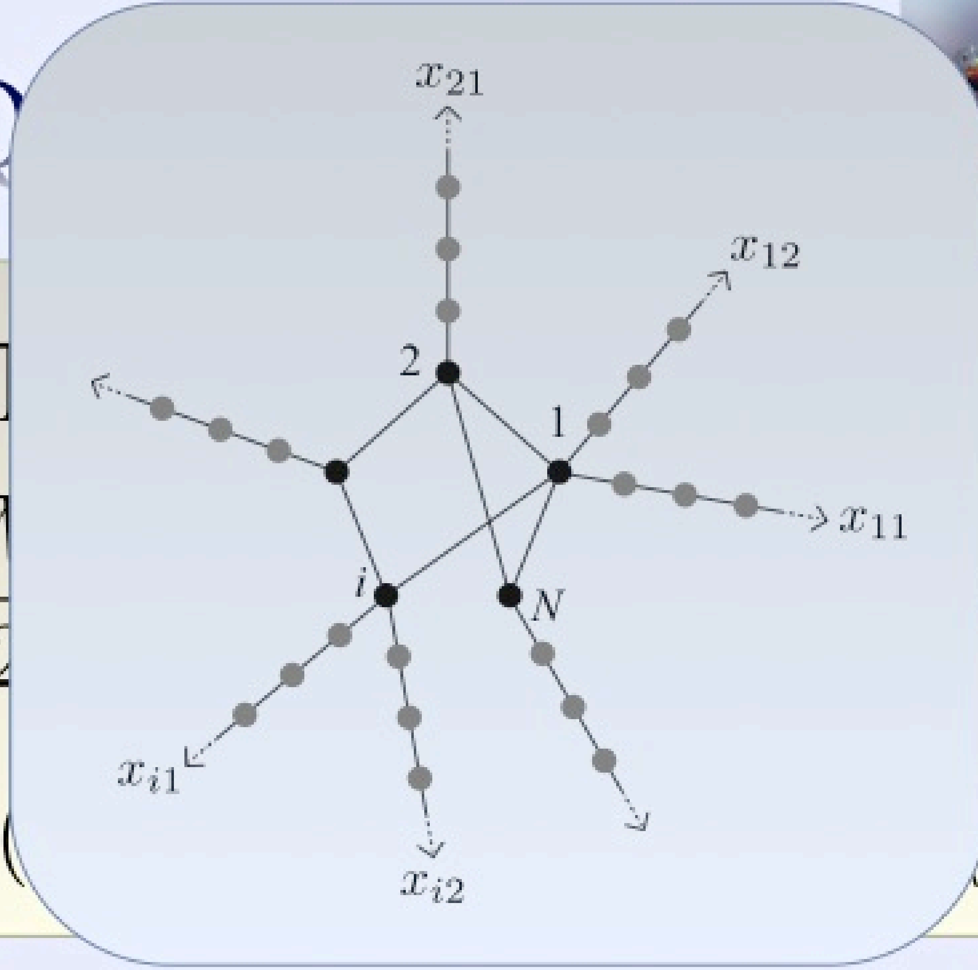
Open Q

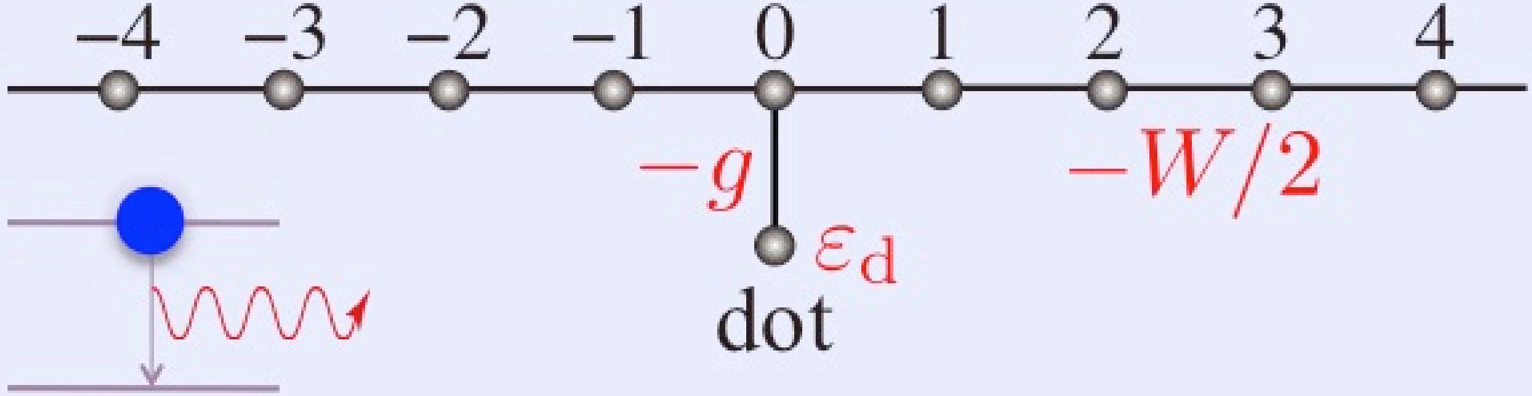


T

$$H = -\frac{W}{2} \ldots \langle x+1|)$$

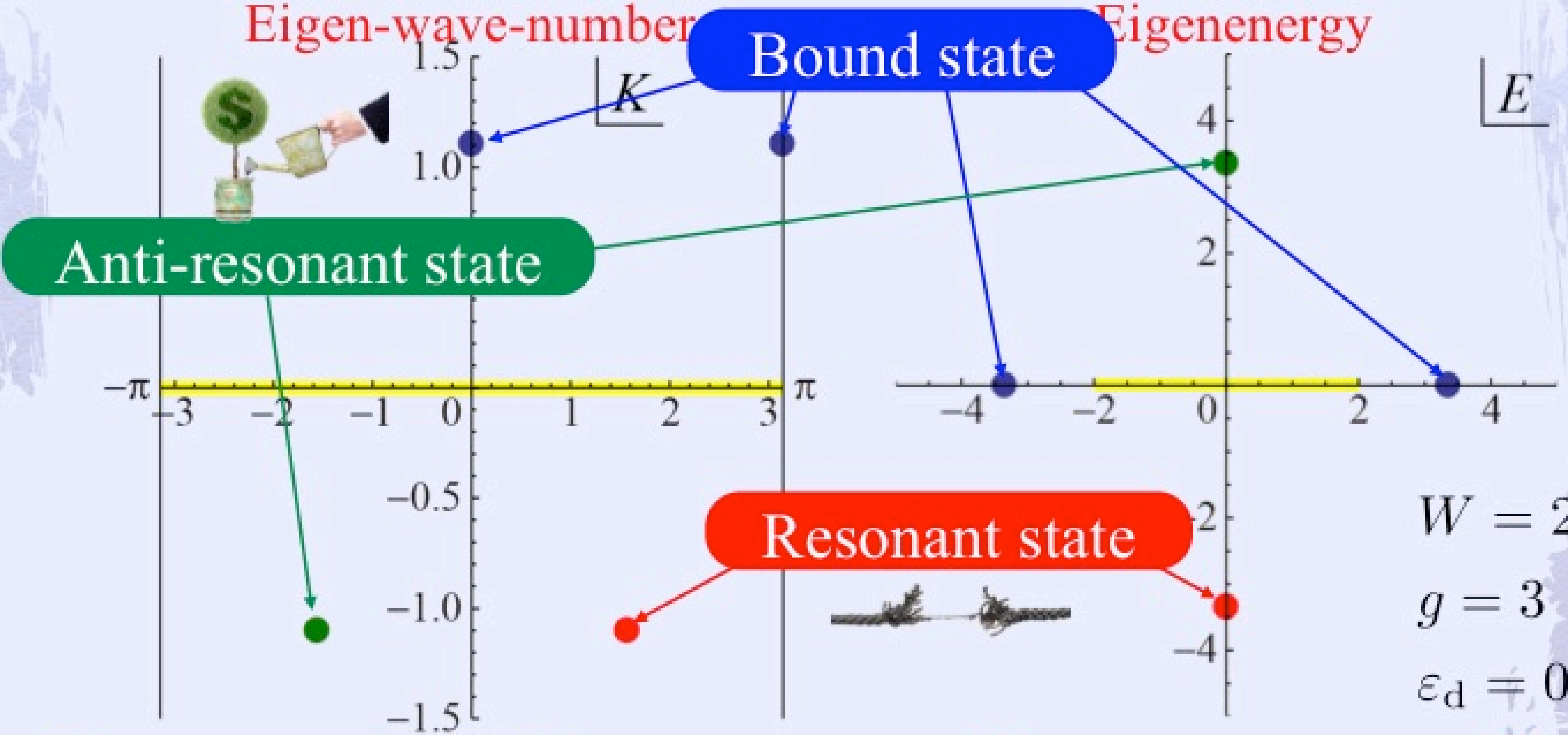
$$-g(\ldots \langle \mathrm{d}|$$





Complex Eigenvalues

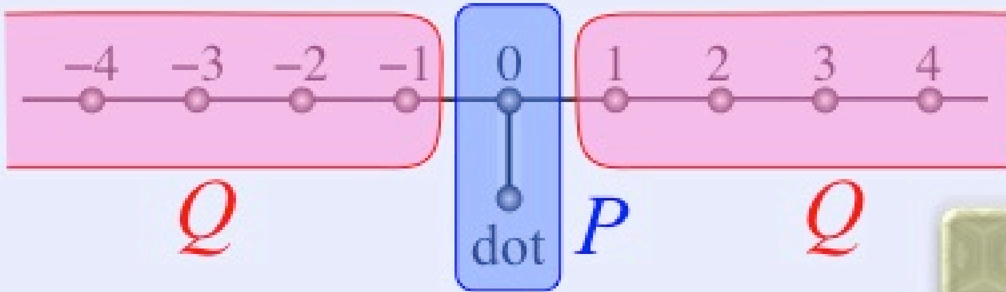
$$E = -2\cos K$$



$$\psi(x) \approx e^{iK|x|}$$

$$\mathrm{Re}\, K_n \gtrless 0 \Leftrightarrow \mathrm{Im}\, E_n \lessgtr 0$$

Transformation of the problem

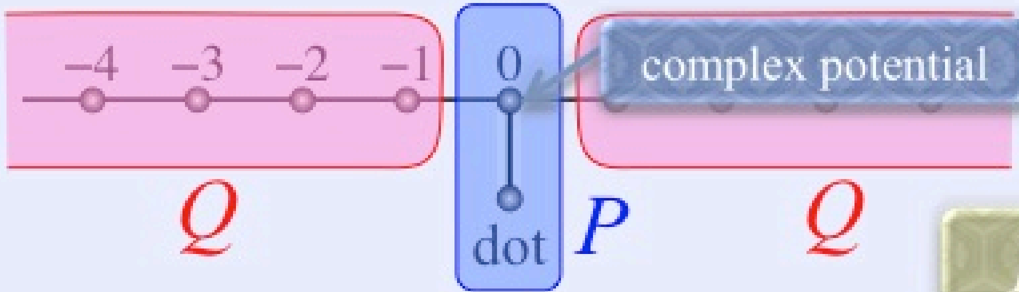


$N = \dim P$

1. ∞-dimensional linear eigenvalue problem $H|\psi\rangle = E|\psi\rangle$
2. $\rightarrow$ N-dimensional non-linear eigenvalue problem
3. $\rightarrow$ $2N$-dimensional linear generalized eigenvalue problem

Feshbach Formalism

∞-dim. → N-dim.



$N = \dim P$

$$H|\psi\rangle = E|\psi\rangle \Rightarrow H_{\text{eff}}(P|\psi\rangle) = E(P|\psi\rangle)$$

$$H_{\text{eff}}(E) = PHP + PHQ\frac{1}{E - QHQ}QHP$$: $N\times N$ matrix

$$\lambda = e^{iK}$$

$$E = -2\cos K = -\lambda - \frac{1}{\lambda}$$

$$[\lambda^2(I - PHQHP) + \lambda PHP + I](P|\psi\rangle) = 0$$

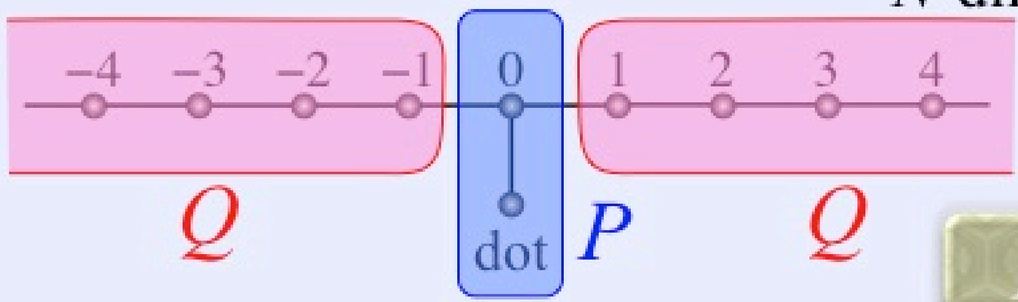
quadratic eigenvalue problem

(S. Klaiman, N. Hatano, J. Chem. Phys. (2011))

Quadratic eigenvalue problem

N-dim. $\to$ $2N$-dim.

$$\lambda = e^{iK}$$



$N = \dim P$

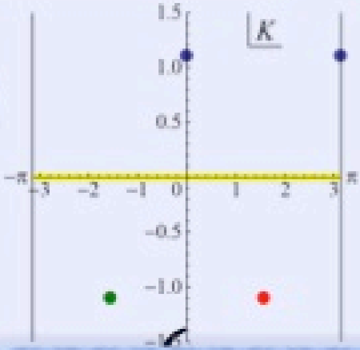
$$\begin{pmatrix} -\lambda I & I \\ I & PHP + \lambda(I - PHQHP) \end{pmatrix} \begin{pmatrix} P|\psi\rangle \\ \lambda P|\psi\rangle \end{pmatrix} = 0$$

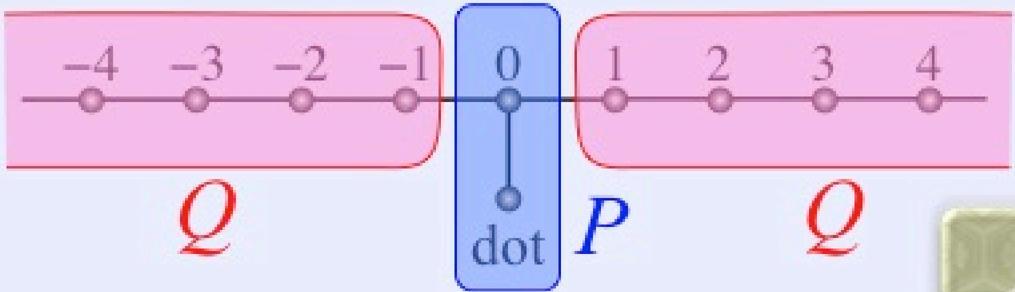
$[\lambda$ $$(A - \lambda B)\,|\Psi\rangle\rangle = 0$$: $2N\times 2N$ matrix 0

$$A = \begin{pmatrix} 0 & I \\ I & PHP \end{pmatrix} \qquad B = \begin{pmatrix} I & 0 \\ 0 & PHQHP - I \end{pmatrix} \qquad |\Psi\rangle\rangle = \begin{pmatrix} P|\psi\rangle \\ \lambda P|\psi\rangle \end{pmatrix}$$

Quadratic eigenvalue problem

$$\lambda = e^{iK}$$





$$N = \dim P$$

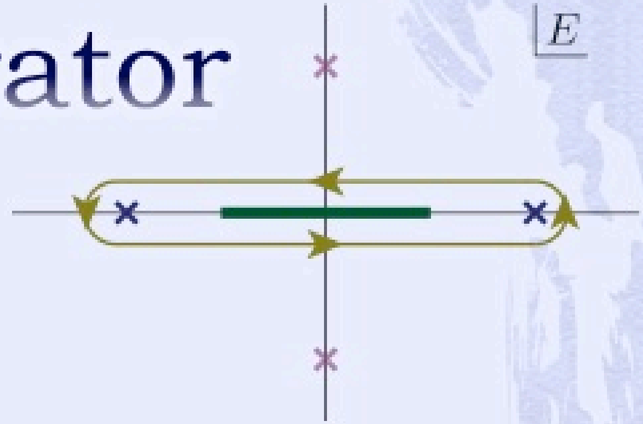
Time-reversal symmetric form

$$(A + \lambda B)|\Psi\rangle = 0$$

$$P\frac{1}{E-H}P = \sum_{n=1}^{2N} P|\psi_n\rangle \frac{\lambda\lambda_n}{\lambda - \lambda_n}\langle\tilde{\psi}_n|P$$

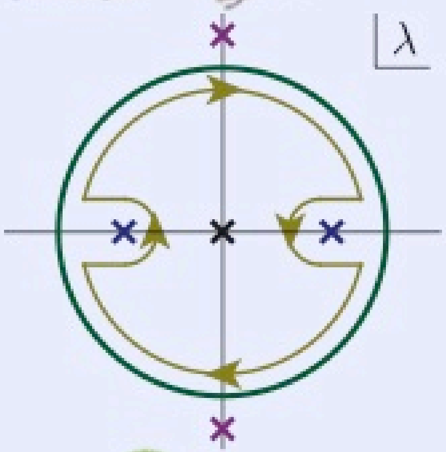
Time evolution operator

N. Hatano, G. Ordonez, arXiv:1405.6683v1



$$Pe^{-iHt}P = \int dE e^{-iEt} P \frac{1}{E-H} P$$

$$= \sum_{n=1}^{2N} \int d\lambda \left(-\lambda + \frac{1}{\lambda}\right) e^{i(\lambda+\lambda^{-1})t} |\psi_n\rangle \frac{\lambda_n}{\lambda - \lambda_n} \langle\tilde{\psi}_n|$$



$$\lambda = e^{iK}$$

$$E = -2\cos K = -\lambda - \frac{1}{\lambda}$$

Time-reversal symmetric form

Time evolution operator

N. Hatano, G. Ordonez, arXiv:1405.6683v1

$$Pe^{-iHt}P = \sum_{n=1}^{2N} \int d\lambda \left(-\lambda + \frac{1}{\lambda}\right) e^{i(\lambda+\lambda^{-1})t} |\psi_n\rangle \frac{\lambda_n}{\lambda - \lambda_n} \langle\tilde{\psi}_n|$$

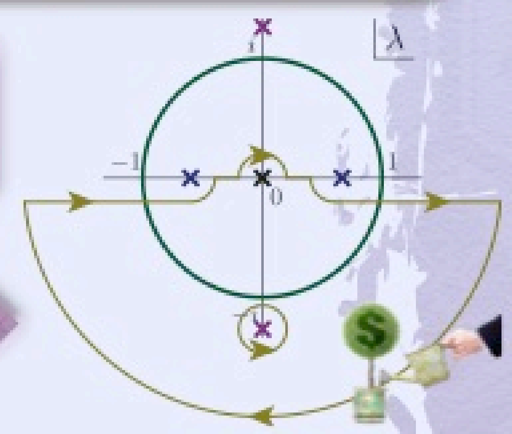
$$= \sum_{n=1}^{2N} \int d\lambda \left(-\lambda + \frac{1}{\lambda}\right) e^{i(\lambda+\lambda^{-1})t} |\psi_n\rangle \frac{\lambda_n}{\lambda - \lambda_n} \langle\tilde{\psi}_n|$$

Evol. from initial cond. → res. st.
Decay

Evol. to terminal cond. → anti-res. st.
Growth

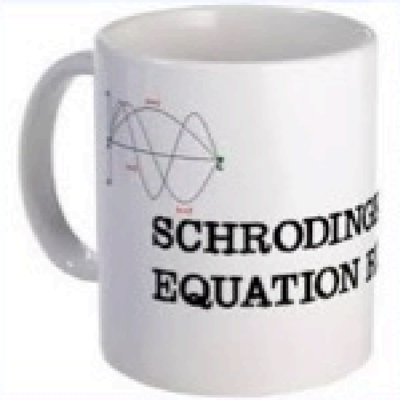
$t > 0$: Chooses res. st.

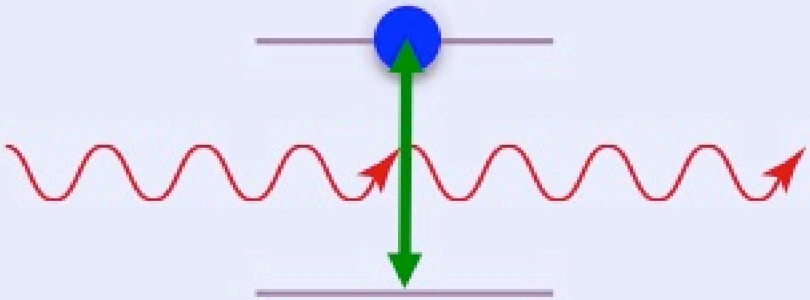
$t < 0$: Chooses anti-res. st.



Time-reversal symmetry breaking

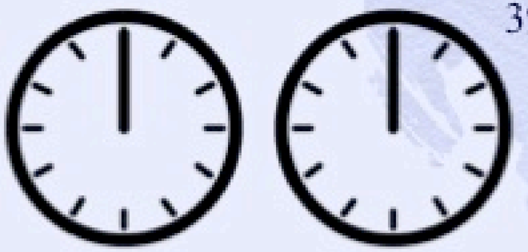
$$i\frac{d}{dt}\psi = H\psi$$

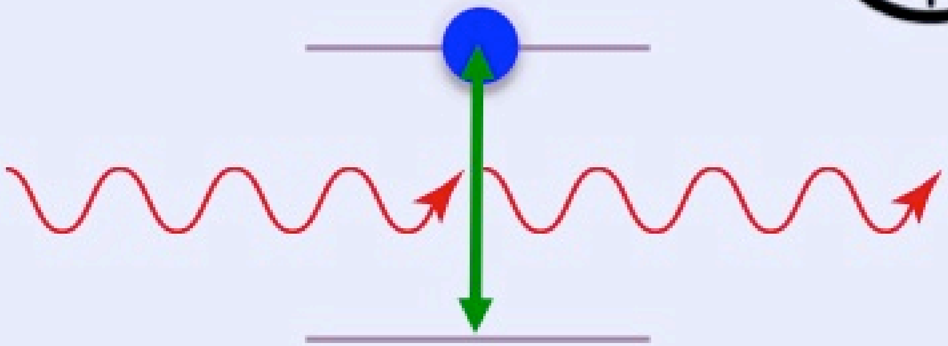




Intrinsic reason	**Extrinsic reason**
Decaying solution (resonant state)	Choice of an initial condition
Growing solution (anti-resonant state)	Choice of a terminal condition

Today’s Message



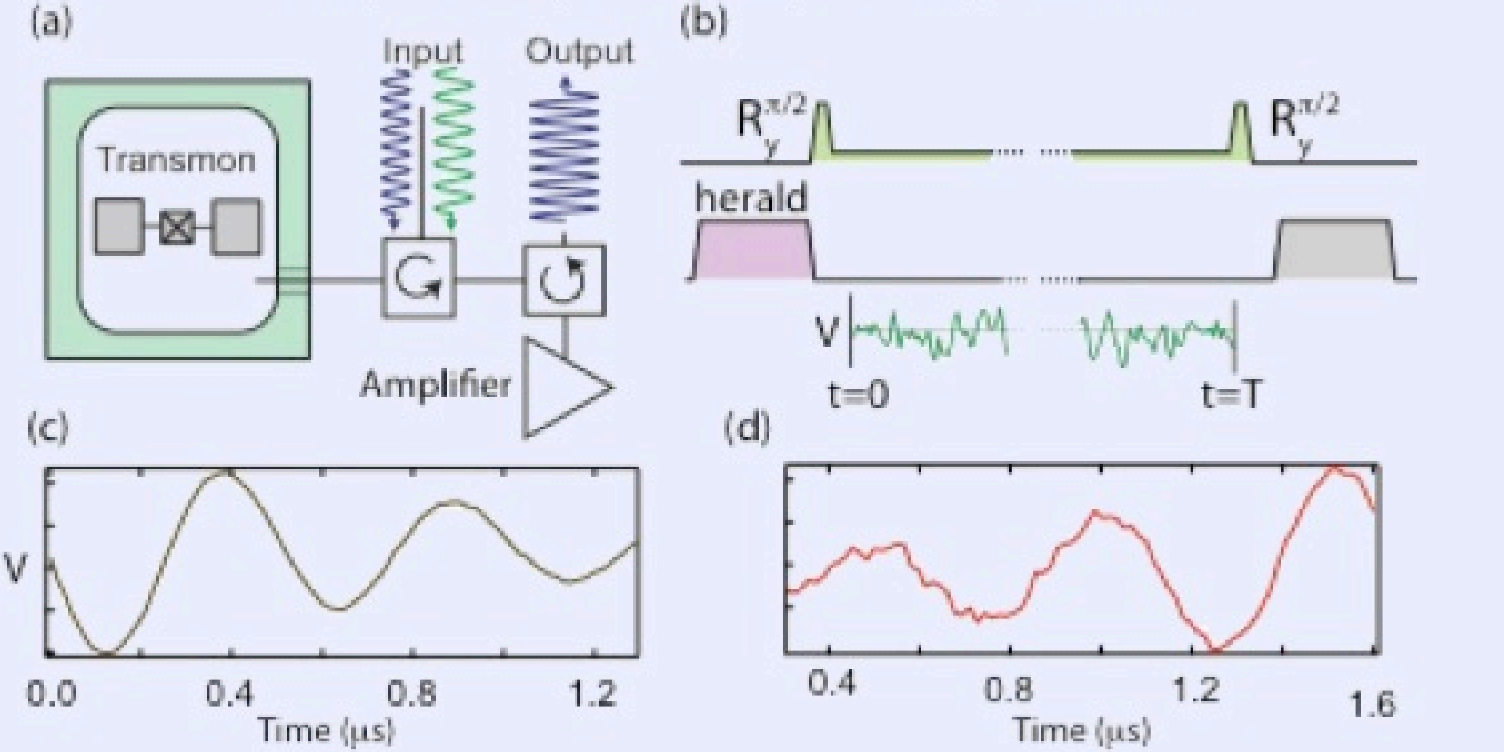


- The excited state decays in the time evolution from an initial condition.
- The excited state grows in the time evolution towards a terminal condition.


Thanks!
For your Time

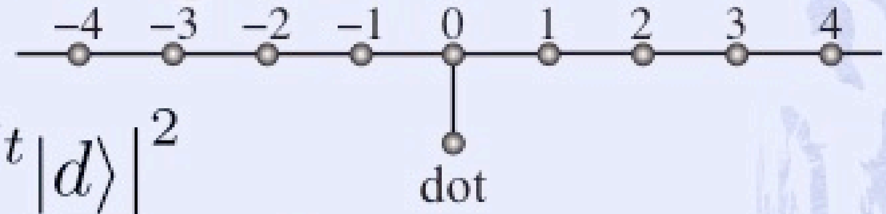
Experiments!

D. Tan, S. Weber, K. Mølmer, I. Siddiqi, K. Murch

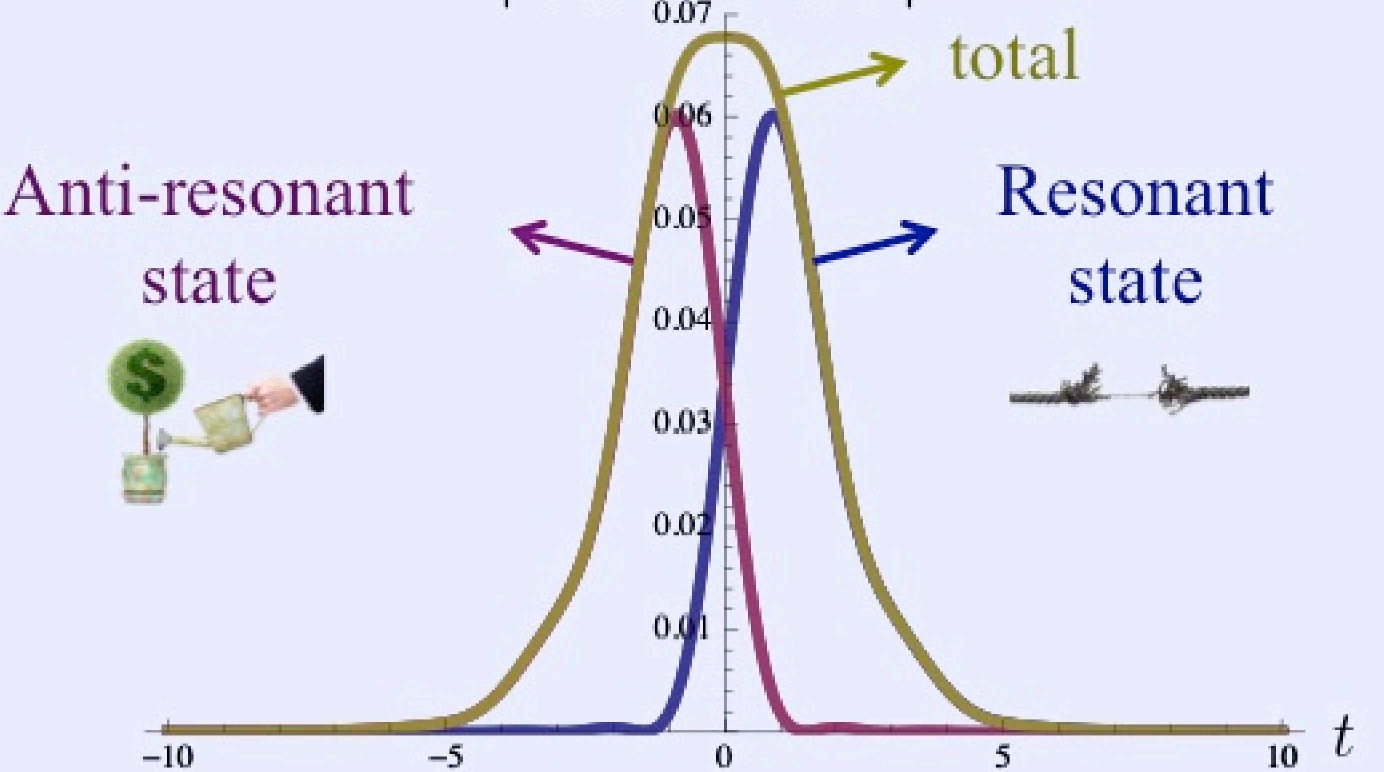


Initial condition problem	Terminal condition problem
Decaying solution	Growing solution

Survival probability

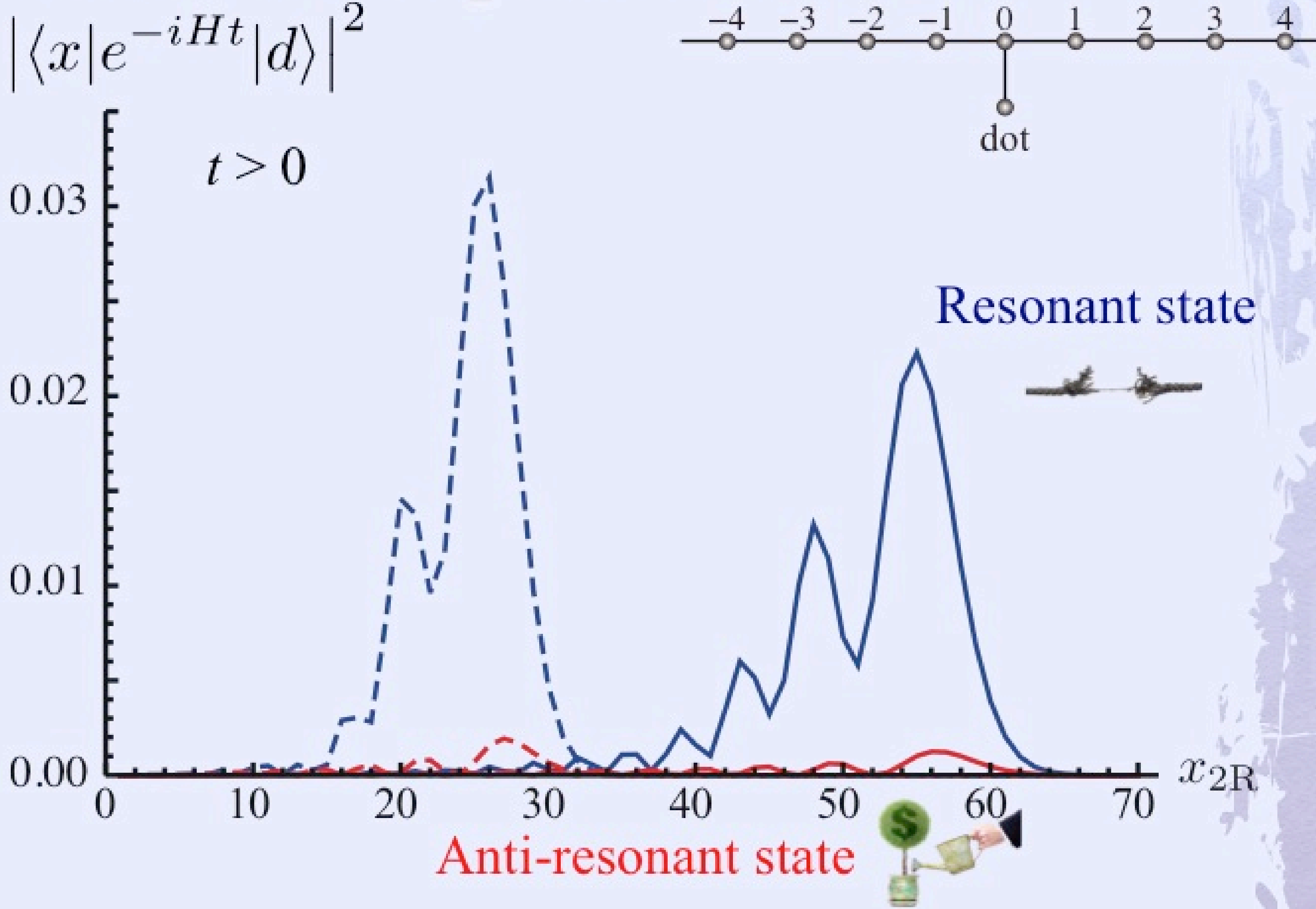


$$\left|\langle d|e^{-iHt}|d\rangle\right|^2$$



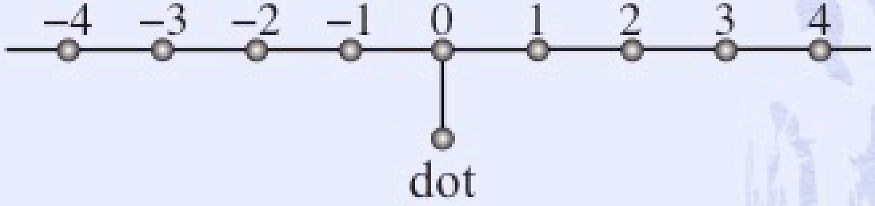
Time-reversal symmetry is strongly broken within the “Zeno” time!

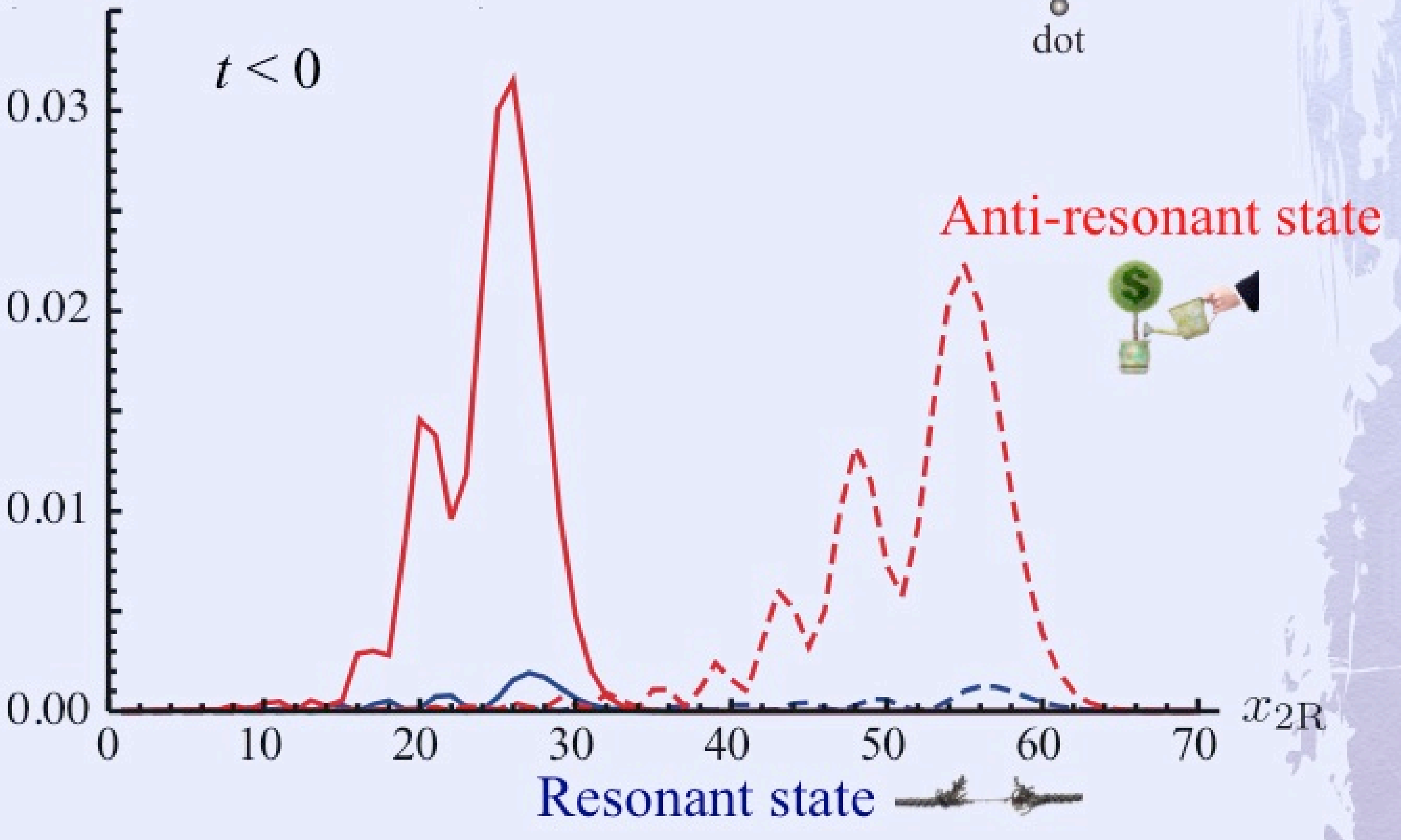
Position representation



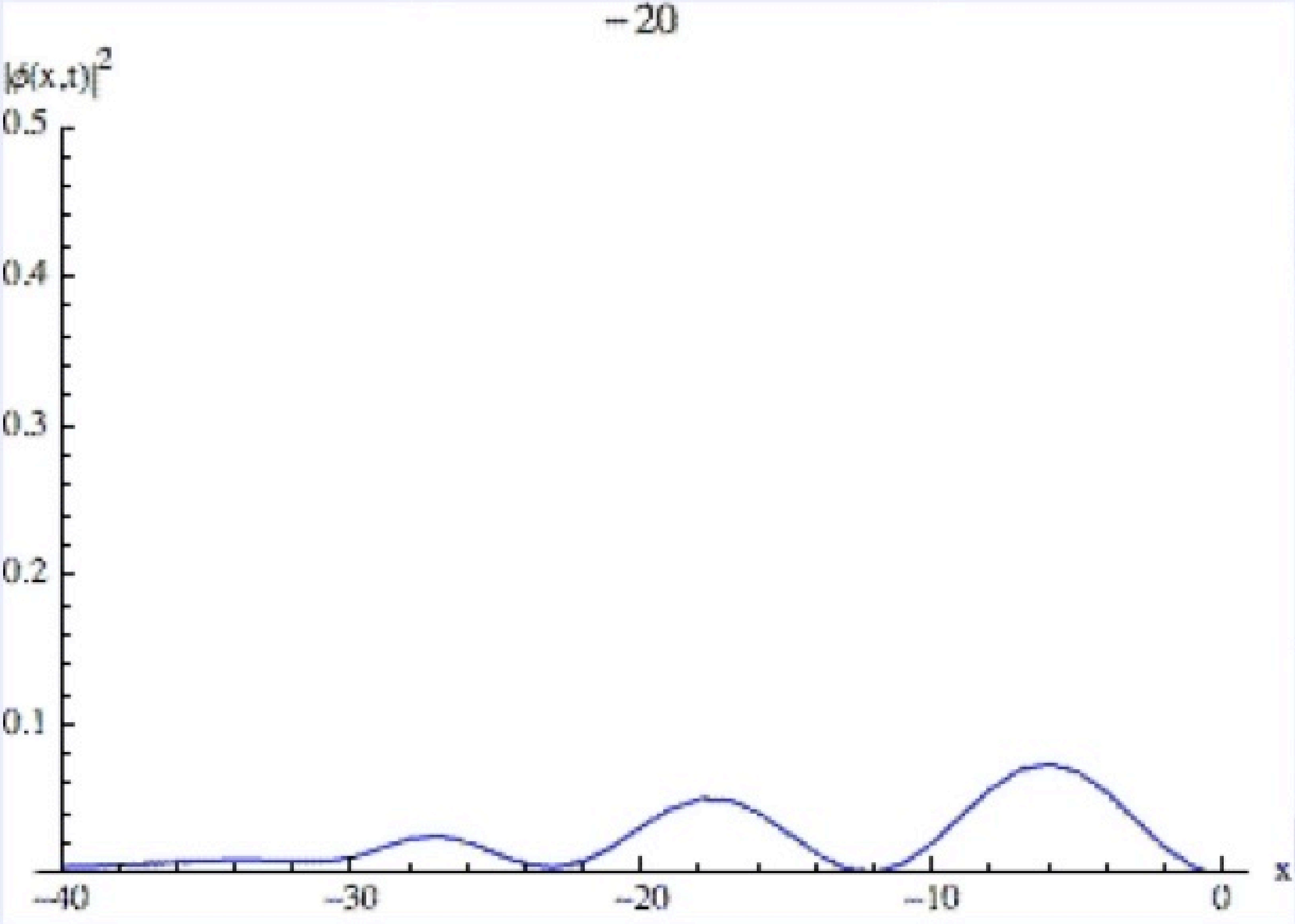
Position representation

$\left|\langle x|e^{-iHt}|d\rangle\right|^2$

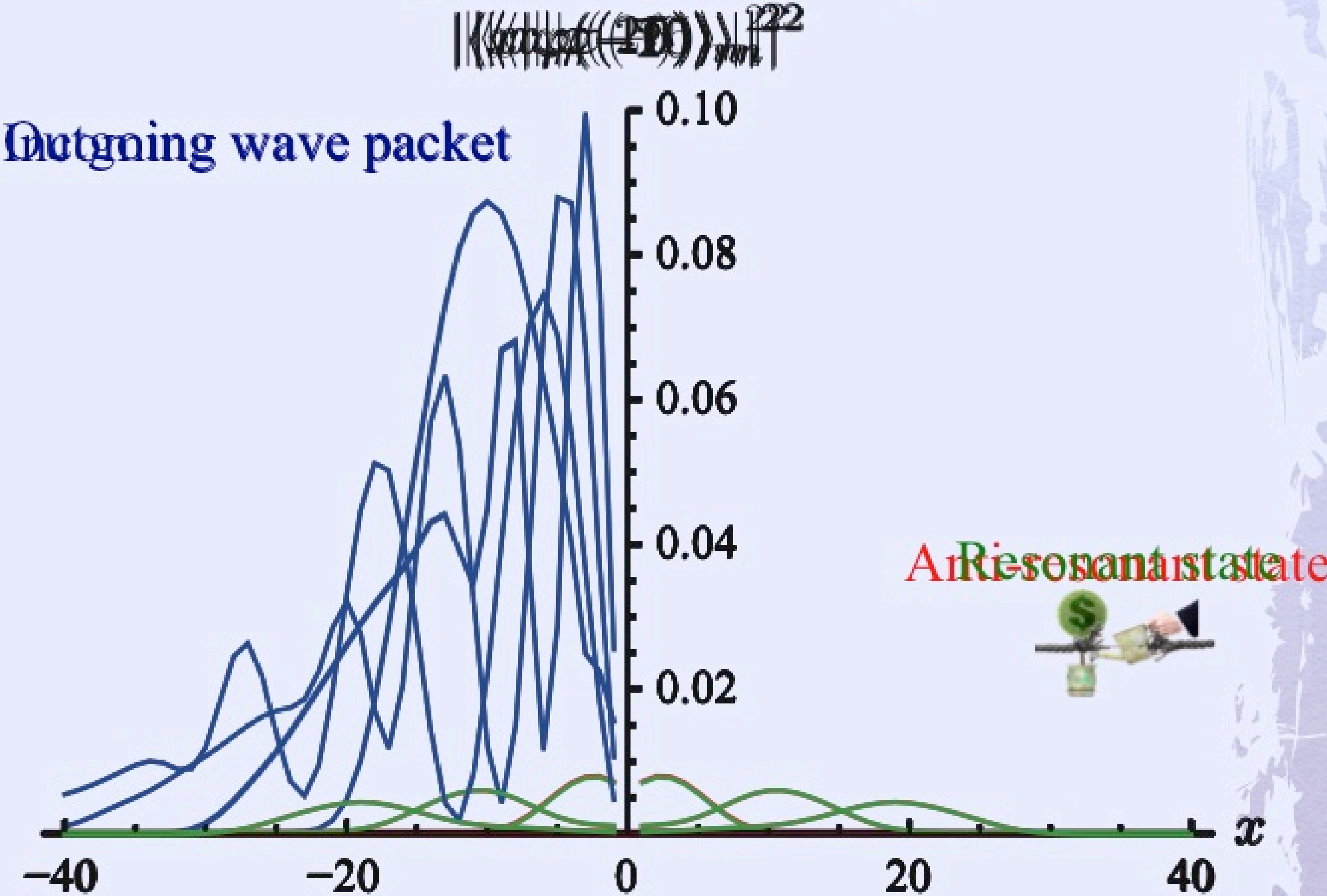




Scattering of a wave packet



Scattering of a wave packet



Liouville-von Neumann eq.

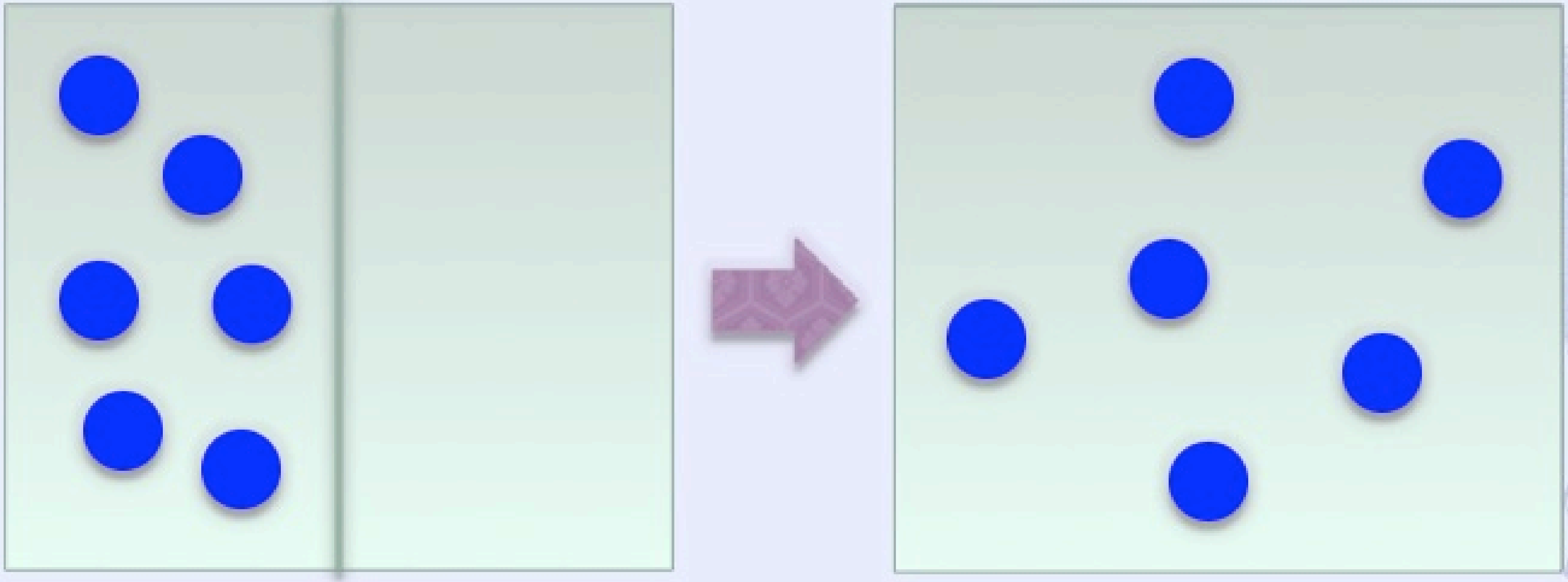
$$i\frac{d}{dt}\rho = [H, \rho]$$

ρ: density matrix

Entropy: $S(t) = -\operatorname{Tr}\rho(t)\ln\rho(t)$

Thremodynamical arrow of time?

Coarse-graining?



small prob.

large prob.

Thremodynamical arrow of time?

Liouville-von Neumann eq.

$$i\frac{d}{dt}\rho = [H, \rho]$$

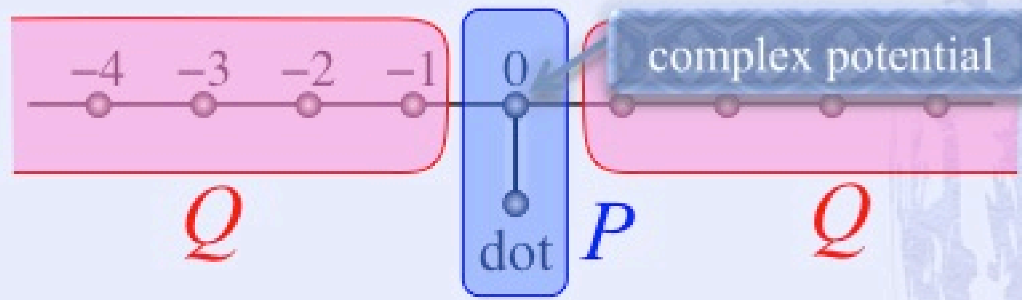
ρ: density matrix

Entropy: $S(t) = -\,\mathrm{Tr}\,\rho(t)\ln\rho(t)$

Thremodynamical arrow of time?

Quadratic eigenvalue problem

Green's function $\dfrac{1}{E-H}$

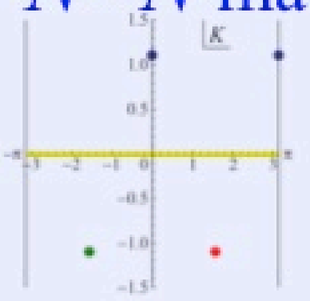


$$P\frac{1}{E-H}P = P\frac{1}{E-H_{\mathrm{eff}}(E)}P$$

$$H_{\mathrm{eff}}(E) = PHP + PHQ\frac{1}{E-QHQ}QHP$$

$N \times N$ matrix

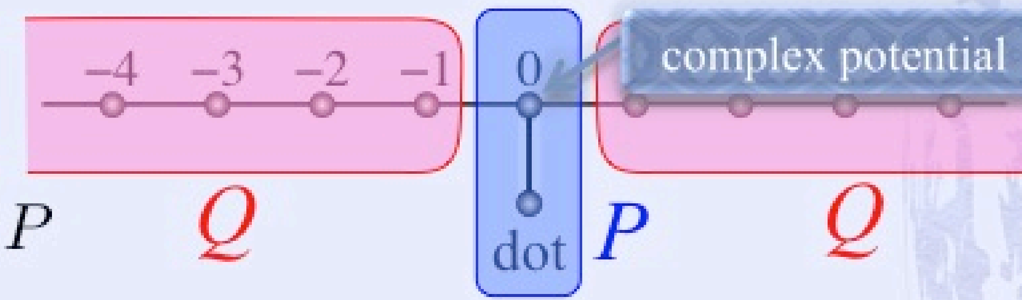
$N = \dim P$



$$(EI - H_{\mathrm{eff}}(E))|\psi\rangle = 0$$

Quadratic eigenvalue problem

$N = \dim P$



$$P\frac{1}{E-H}P = P\frac{1}{E-H_{\rm eff}(E)}P$$

$$H_{\rm eff}(E) = PHP + PHQ\frac{1}{E-QHQ}QHP$$: $N \times N$ matrix

$$\lambda = e^{iK}$$

$$E = -2\cos K = -\lambda - \frac{1}{\lambda}$$

$$\left[\lambda^2\left(I - PHQHP\right) + \lambda PHP + I\right]|\psi\rangle = 0$$

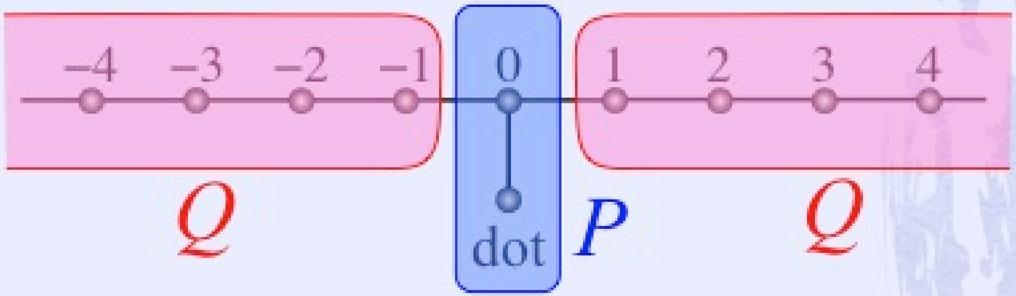
$$(EI - H_{\rm eff}(E))|\psi\rangle = 0$$

Qu lem

Quadratic eigenvalue problem

$N = \dim P$

$\lambda = e^{iK}$



$$\left[\lambda^2 (I - PHQHP) + \lambda PHP + I\right] |\psi\rangle = 0$$

linearization

$$\begin{pmatrix} -\lambda I & I \\ I & PHP + \lambda(I - PHQHP) \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ \lambda|\psi\rangle \end{pmatrix} = 0$$

$$(A - \lambda B)\,|\Psi\rangle\rangle = 0$$

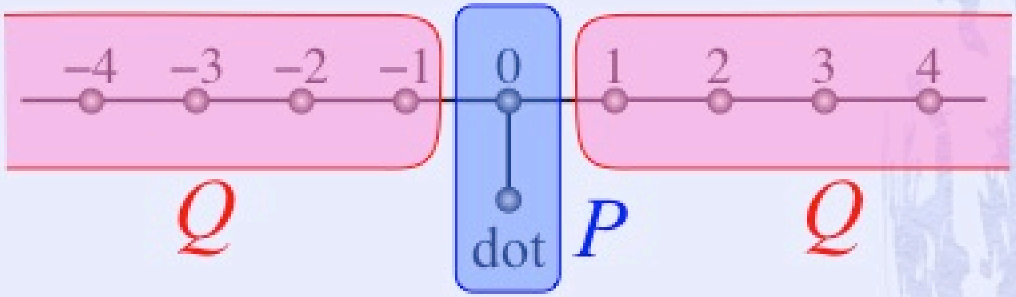
: $2N \times 2N$ matrix

$$A = \begin{pmatrix} 0 & I \\ I & PHP \end{pmatrix} \qquad B = \begin{pmatrix} I & 0 \\ 0 & I - PHQHP \end{pmatrix} \qquad |\Psi\rangle = \begin{pmatrix} |\psi\rangle \\ \lambda|\psi\rangle \end{pmatrix}$$

Quadratic eigenvalue problem

$N = \dim P$

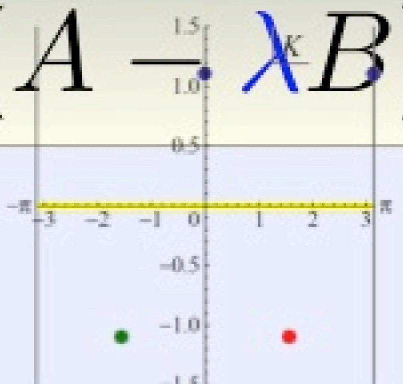
$\lambda = e^{iK}$

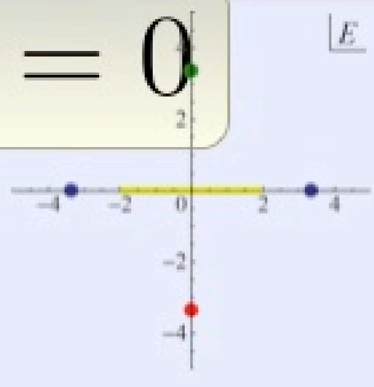


$$\rightarrow \left\{ \lambda_n, |\psi_n\rangle, \langle\tilde{\psi}_n| = |\psi_n\rangle^T \;\middle|\; 1 \le n \le 2N \right\}$$

All $2N$ poles of the S matrix

$$(A - \lambda B)\,|\Psi\rangle\!\rangle = 0$$





Green's function expansion

$$P\frac{1}{E-H}P = \frac{1}{E-H_{\mathrm{eff}}(E)}$$

$$= -\lambda \begin{pmatrix} I & 0 \end{pmatrix} \frac{1}{A-\lambda B} \begin{pmatrix} 0 \\ I \end{pmatrix}$$

$$= \sum_{n=1}^{2N} |\psi_n\rangle \frac{\lambda\lambda_n}{\lambda-\lambda_n} \langle\psi_n|$$

Feshbach Formalism

I. Prigogine and T. Petrosky; I. Rotter et al.

$$P\frac{1}{E-H}P=\frac{1}{E-H_{\mathrm{eff}}(E)}$$

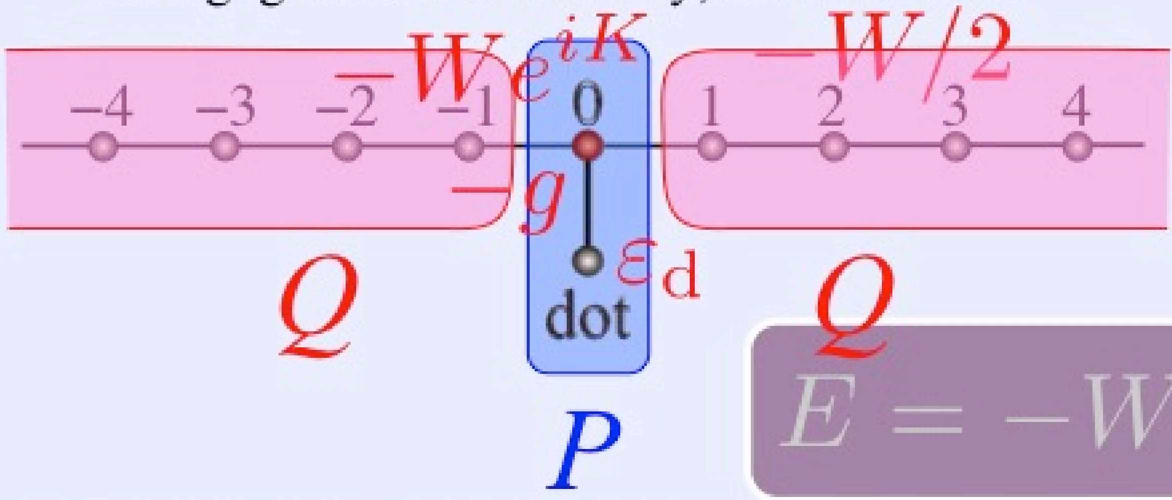
$$\left(PHP+PHQ\frac{1}{E-QHQ}QHP\right)(P|\psi\rangle)=E(P|\psi\rangle)$$

No Approximation. Simple Algebra.

$$H_{\mathrm{eff}}(E)(P|\psi\rangle)=E(P|\psi\rangle)$$

Feshbach Formalism

I. Prigogine and T. Petrosky; I. Rotter et al.



$$E = -W \cos K$$

$$\left(PHP + PHQ \frac{1}{E - QHQ} QHP \right) (P|\psi\rangle) = E(P|\psi\rangle)$$

$$\left[\begin{pmatrix} -We^{iK} & -g \\ -g & \varepsilon_{\mathrm{d}} \end{pmatrix} - EI \right] \begin{pmatrix} |0\rangle \\ |\mathrm{d}\rangle \end{pmatrix} = 0$$

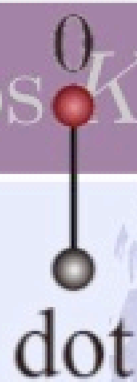
Quadratic Eigenvalue Problem

S. Klaiman, N. Hatano, J. Chem. Phys. **134** (2011) 154111

$$\cos K$$

$$\lambda \equiv e^{iK}$$

$$E = -\frac{W}{2}\left(\lambda + \frac{1}{\lambda}\right)$$



$$\left[-\lambda\frac{W}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & -g\\ -g & \varepsilon_{\mathrm{d}}\end{pmatrix} + \frac{W}{2\lambda}I\right]\begin{pmatrix}|0\rangle\\ |\mathrm{d}\rangle\end{pmatrix} = 0$$

$$\left[\begin{pmatrix}-We^{iK} & -g\\ -g & \varepsilon_{\mathrm{d}}\end{pmatrix} - EI\right]\begin{pmatrix}|0\rangle\\ |\mathrm{d}\rangle\end{pmatrix} = 0$$

Quadratic Eigenvalue Problem

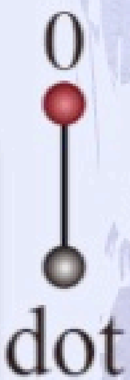
S. Klaiman, N. Hatano, J. Chem. Phys. **134** (2011) 154111

$$(-\lambda^2\sigma^z + \lambda H_\mathrm{d} + I)|\psi\rangle = 0$$

$$\begin{pmatrix} -\lambda I & I \\ I & H_\mathrm{d} - \lambda\sigma^z \end{pmatrix}\begin{pmatrix} |\psi\rangle \\ \lambda|\psi\rangle \end{pmatrix} = 0$$

$$\left[-\lambda\frac{W}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -g \\ -g & \varepsilon_\mathrm{d} \end{pmatrix} + \frac{W}{2\lambda}I\right]\begin{pmatrix} |0\rangle \\ |\mathrm{d}\rangle \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 0 & I \\ I & H_\mathrm{d} \end{pmatrix} - \lambda\begin{pmatrix} I & 0 \\ 0 & \sigma^z \end{pmatrix}\right]|\Psi(\lambda)\rangle\rangle = 0$$



Green's function

$$\left[\begin{pmatrix}0 & I\\ I & H_\mathrm{d}\end{pmatrix} - \lambda\begin{pmatrix}I & 0\\ 0 & \sigma^z\end{pmatrix}\right]|\Psi(\lambda)\rangle\rangle = 0$$

$$(A-\lambda B)|\Psi\rangle\rangle = 0$$

$$P\frac{1}{E-H}P = \frac{1}{E-H_\mathrm{eff}(E)}$$

$$= -\lambda\begin{pmatrix}I & 0\end{pmatrix}\frac{1}{A-\lambda B}\begin{pmatrix}0\\ I\end{pmatrix}$$

Green's function expansion

$$P\frac{1}{E-H}P = \frac{1}{E - H_{\mathrm{eff}}(E)}$$

$$= \sum_{n=1}^{2N} |\psi_n\rangle \frac{\lambda\lambda_n}{\lambda - \lambda_n} \langle\psi_n|$$